

Interval Values for Multicriteria Cooperative Games

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Abstract In this paper we consider multicriteria interval games. The importance of multi-objectives follows from applications to real world. We consider interval valued games and extend some classical solutions for cooperative games to this new class in multicriteria situations.

Keywords Multicriteria situations, interval valued games, cooperative games, core

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1. Introduction

Cooperative game theory is concerned with coalitions or groups of players who coordinate their actions and put together their winnings. For each set S of players, $v(S)$ indicates the amount they can gain if they form the coalition S , excluding the other players. A problem is how to divide the extra-earning among the coalition members. For more information about cooperative games see Owen (1995).

In applications to real life (particularly in the economic field) rewards are not known precisely but they can vary in an interval. In Moore (1979) an analysis on intervals was introduced and this work inspired the paper of Yager and Kreinovic (2000). Cooperative games with interval values, called *cooperative interval games*, were introduced in Branzei et al. (2003) and studied in Alparslan et al. (2009), Branzei et al. (2008), too. Carpenete et al. (2008) studied how to associate a coalitional interval game to each strategic game.

We think that interval analysis is a new, growing branch of applied mathematics. The basis of this method is an extension of the concept of “real number”. An interval of real numbers can be seen as a new kind of number, represented by a pair of real numbers, namely its endpoints. We cannot carry out classical arithmetic. So we have to consider a new one and through intervals we will be able to compute bounds on the set of solutions.

Applications of interval methods can be found in mathematical programming, operator equations, geodetic computation, analysis of electrical circuits and the re-entry of an Apollo-type spaceship into the earth’s atmosphere. In studying a cost-benefit analysis of a project, the World Bank measured the profitability of a project by computing the rate of return of a project. Interval methods turned out to be appropriate and

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so it was used by the World Bank. Interval techniques can be successfully applied to the theory of differential equations too. For a deeper study about all these examples, see Moore (1979).

In this paper we present cooperative games where the value of the coalitions is a vector of intervals; this is a way to model multiobjective problems. Multiobjective games in the non-cooperative case were studied in Patrone et al. (2007), Puerto Albandoz et al. (2006), Spapley (1959), Yager et al. (2000) and an interesting axiomatic approach was given in Miglierina et al. (2008). In our paper we consider cooperative multicriteria games which are also known as “cooperative games with vector payoff” or “cooperative multiobjective games”. We consider multiobjective and multiagent games in the cooperative case with interval values.

The concept of coalition in the multiobjective case is the classical one. We will write $\mathfrak{I}(\mathbb{R})$ for the set of non-empty compacts of $\mathbb{R}$; $\mathfrak{I}(\mathbb{R}^m)$ for the set of vectors of dimension m with components in $\mathfrak{I}(\mathbb{R})$; $\mathfrak{I}(\mathbb{R}^{m \times n})$ for the set of matrices of dimension $m \times n$ with elements in $\mathfrak{I}(\mathbb{R})$. A cooperative multicriteria game with interval values is defined as $G = \langle N, v \rangle$, where $N = \{1, 2, \dots, n\}$ is the set of players and the characteristic function is:

$$v : 2^N \rightarrow \mathfrak{I}(\mathbb{R}^m)$$

It associates to each S an m -vector of intervals denoted by $v(S)$, with $v(\emptyset) = 0$ (that is an m -vector of intervals of length zero).

This paper comprises 6 sections: in Section 2 a cooperative approach is considered and some properties about the algebra of intervals are given; in Section 3 we study multiobjective cooperative games with interval values and give some definitions and examples to illustrate some applied problems; Section 4 contains the main results; in Section 5 we study the approximate core for this new class of games. Conclusions and some suggestions for open problems are given in Section 6.

2. Notation and preliminary results about classical cooperative games

A cooperative n -person game in coalitional form and with one criterium is an ordered pair $\langle N, v \rangle$, where $N = \{1, 2, \dots, n\}$ is the set of players and v is the characteristic function, $v : 2^N \rightarrow \mathbb{R}$ which assigns to each non empty coalition S , a real number $v(S) \in \mathbb{R}$, $v(S)$ is called the worth of coalition and $v(\emptyset) = 0$.

If $\langle N, v \rangle$ is a cooperative game with n players, a vector $x \in \mathbb{R}^n$ is called an *imputation* if: (i) $x_i \geq v(\{i\})$, for all $i \in N$ (that is the vector x has the property of individual rationality); (ii) $\sum_{i=1}^n x_i = v(N)$ (that is x is efficient in the sense of Pareto). The set of imputations is denoted $I(v)$.

For a cooperative game $\langle N, v \rangle$ with n players, the *core* is defined as:

$$C(v) = \left\{ x \in I(v) \mid \sum_{i \in S} x_i \geq v(S), \forall S \in 2^N \setminus \{\emptyset\} \right\}$$

Intuitively, if $x \in C(v)$, x is the proposed reward allocation in N , and no coalition $S \neq N$ has any incentive to go out because the amount $x^S = \sum_{i \in S} x_i$ allocated to S is not smaller than the amount $v(S)$ which can be obtained by the coalition.

A map $\lambda : 2^N \setminus \{\emptyset\} \rightarrow \mathbb{R}_+$ is called a *balanced map* if

$$\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) e^S = e^N,$$

where

$$e_i^S = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}.$$

Given $x, y \in \mathbb{R}^k$ we consider the following preferences on $\mathbb{R}^k$:

$$x \geq y \Leftrightarrow x_i \geq y_i \forall i = 1, \dots, k;$$

$$x \geq y \Leftrightarrow x \geq y \text{ and } x \neq y;$$

$$x > y \Leftrightarrow x_i > y_i \forall i = 1, \dots, k.$$

Here we summarize some operations in the algebra of intervals, let us consider non empty compact intervals of real numbers. Let $a, b, c, d \in [0, +\infty)$, $\alpha \in [0, +\infty)$ with $a \leq b$; $c \leq d$. Let us put:

$$(i) [a, b] = [c, d] \Leftrightarrow a = c \text{ and } b = d;$$

$$(ii) [a, b] + [c, d] = [a + c, b + d];$$

$$(iii) \alpha[a, b] = [\alpha a, \alpha b];$$

$$(iv) [a, b] \geq [c, d] \Leftrightarrow a \geq c \text{ and } b \geq d;$$

$$(v) [a, b] > [c, d] \Leftrightarrow a > c \text{ and } b > d \text{ (for more details see Moore 1979)}.$$

To keep notation easy we use the symbols $\geq, >$ for relations between vectors in $\mathbb{R}^k$, for relations between non empty compact intervals of real numbers, and for relations between m -dimensional vectors of real compact intervals. The specific meaning is clear from the context.

3. Multiobjective cooperative games with interval values

From here on, unless otherwise specified, we consider only multiobjective cooperative games with interval values. We consider games with n players and m objectives. Let N be the set of players and $v : 2^N \rightarrow \mathfrak{I}(\mathbb{R}^m)$ be the characteristic function of the game G . It assigns to each coalition $S \in 2^N \setminus \{\emptyset\}$, m closed intervals

$$v(S) = \begin{pmatrix} v_1(S) \\ v_2(S) \\ \vdots \\ v_m(S) \end{pmatrix} = \begin{pmatrix} [\underline{v}_1(S), \bar{v}_1(S)] \\ [\underline{v}_2(S), \bar{v}_2(S)] \\ \vdots \\ [\underline{v}_m(S), \bar{v}_m(S)] \end{pmatrix},$$

where $\underline{v}_j(S)$ is the left endpoint and $\bar{v}_j(S)$ is the right endpoint of the interval $v_j(S)$ associated to coalition S and objective j and $v(\emptyset) = 0$.

Let $X \in \mathfrak{S}(\mathbb{R}^{m \times n})$, we write:

$$X = \begin{pmatrix} X_1^1 & X_1^2 & \cdots & X_1^n \\ X_2^1 & X_2^2 & \cdots & X_2^n \\ \vdots & \vdots & \ddots & \vdots \\ X_m^1 & X_m^2 & \cdots & X_m^n \end{pmatrix} = \begin{pmatrix} [a_1^1, b_1^1] & [a_1^2, b_1^2] & \cdots & [a_1^n, b_1^n] \\ [a_2^1, b_2^1] & [a_2^2, b_2^2] & \cdots & [a_2^n, b_2^n] \\ \vdots & \vdots & \ddots & \vdots \\ [a_m^1, b_m^1] & [a_m^2, b_m^2] & \cdots & [a_m^n, b_m^n] \end{pmatrix},$$

where $[a_j^i, b_j^i]$ is the interval relative to player i and objective j . Sometimes we write: $[a_j^i, b_j^i] = [\underline{v}_j^i, \bar{v}_j^i]$ with obvious interpretation.

When we write X^i , we wish to stress the component of player i :

$$X^i = \begin{pmatrix} [a_1^i, b_1^i] \\ [a_2^i, b_2^i] \\ \vdots \\ [a_m^i, b_m^i] \end{pmatrix}$$

If we write X_j we wish to stress the component relative to objective j :

$$X_j = ([a_j^1, b_j^1], [a_j^2, b_j^2], \dots, [a_j^n, b_j^n])$$

Example 1. Let G be the cooperative game below:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	$[2,3]$	$[3,4]$	$[4,5]$	$[6,8]$	$[6,8]$	$[9,11]$	$[12,15]$
	$[2,3]$	$[4,5]$	$[1,2]$	$[7,9]$	$[3,5]$	$[4,6]$	$[10,13]$

By this notation we mean $v(\{1\}) = \begin{pmatrix} [2,3] \\ [2,3] \end{pmatrix}$, $v(\{2,3\}) = \begin{pmatrix} [9,11] \\ [4,6] \end{pmatrix}$ and so on.

Definition 1. A preimputation of the game G is a matrix $X = (X_j^i) \in \mathfrak{S}(\mathbb{R}^{m \times n})$ such that $\sum_{i=1}^n X_j^i = v_j(N)$, $\forall j = 1, \dots, m$. We will write $I^*(v)$ for the preimputation set of the game G .

We mean $X^S = \sum_{i \in S} X^i = \sum_{i \in S} \begin{pmatrix} X_1^i \\ \vdots \\ X_m^i \end{pmatrix}$. In the previous example, if $S = \{1, 2\}$, then

$$\begin{aligned} X^S &= X^{\{1,2\}} = \sum_{i \in \{1,2\}} X^i = X^1 + X^2 = \begin{pmatrix} X_1^1 \\ X_2^1 \end{pmatrix} + \begin{pmatrix} X_1^2 \\ X_2^2 \end{pmatrix} = \\ &= \begin{pmatrix} [2,3] \\ [2,3] \end{pmatrix} + \begin{pmatrix} [3,4] \\ [4,5] \end{pmatrix} = \begin{pmatrix} [5,7] \\ [6,8] \end{pmatrix}. \end{aligned}$$

We say that the game $\langle N, v \rangle$ has the *monotonic property* if

$$v(S) \geq v(T), \forall T, S \in 2^N \setminus \{\emptyset\}, T \subset S.$$

We will say that a cooperative game has the property of *weak-superadditivity* if

$$v(S) \geq \sum_{i \in S} v(i), \quad \forall S \in 2^N \setminus \{\emptyset\}.$$

Definition 2. A preimputation X is an imputation if $X^i \geq v(\{i\})$, $\forall i \in N$. We will call $I(v)$ the imputation set.

An imputation for the game in the Example 1 is the following:

$$X = \left(\begin{array}{ccc} [3, 4] & [4, 5] & [5, 6] \\ [2, 3] & [4, 5] & [4, 5] \end{array} \right)$$

Definition 3. Let us define the core for the game $\langle N, v \rangle$ in the following way:

$$C(v) = \{X \in I^*(v) \mid X^S \geq v(S), \forall S \in 2^N \setminus \{\emptyset\}\}$$

Example 2. This cooperative game is a generalization of the “Sinergia entre empresas” in Puerto Albandoz et al. (2006). There is an agricultural society (player I) dedicated to fruit and vegetable production, an industry comprising a canning industry (player II) and a service enterprise (player III). This last one is also dedicated to marketing and distribution of food.

Each year the society obtains a net benefit whose value is between 20 million euros and 30 million euros, moreover a number between 200 and 300 employees is financed by the Administration; the industry obtains each year a value between 30 and 40 million euros and moreover a number between 400 and 500 employees is financed and the service enterprise obtains each year a benefit between 40 and 50 million euros and a number between 100 and 200 employees is financed by the Administration.

The Administration has a project to improve the cooperation between enterprises. If they follow the proposed plans and they make a total cooperation, they can gain a benefit between 120 and 150 million euros and they can obtain a number between 1000 and 1300 subsidized work-places.

The management offices of these three enterprises studied their benefits when the cooperation is between two of these three enterprises. The society and the industry (I and II) can increase their total minimum benefits by at least 10 million euros and have 100 workers; the cooperation between society and enterprise (I and III) improves neither benefits nor worker subsidies, on the contrary the maximum benefit and the maximum number of subsidized workers decrease. The industry and the enterprise (II and III) increase their benefits with at least 20 million euros and at most 10 million euros but they can loose at least 100 work-place subsidies and at most 200.

In this game we consider three players: the society, the industry and the enterprise. We have two objectives: the net benefit (in million euros) and the total number of subsidies for workers (one hundred work-places which cost from Administration). So if we call S the coalitions and $v(S)$ the benefit which every coalition can obtain, we can write the game as:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	$[2,3]$	$[3,4]$	$[4,5]$	$[6,7]$	$[6,7]$	$[9,10]$	$[12,15]$
	$[2,3]$	$[4,5]$	$[1,2]$	$[7,8]$	$[3,4]$	$[4,5]$	$[10,13]$

Let $X = \left(\begin{array}{ccc} [3,6] & [4,4] & [5,5] \\ [3,5] & [5,6] & [2,2] \end{array} \right)$, $X \in I(v)$. In fact,

$$\sum_{i=1}^3 X_1^i = v_1(N) = [3,6] + [4,4] + [5,5] = [12,15],$$

$$\sum_{i=1}^3 X_2^i = v_2(N) = [3,5] + [5,6] + [2,2] = [10,13],$$

$$X^1 = \left(\begin{array}{c} [3,6] \\ [3,5] \end{array} \right) \geq \left(\begin{array}{c} [2,3] \\ [2,3] \end{array} \right) = v(\{1\}),$$

$$X^2 = \left(\begin{array}{c} [4,4] \\ [5,6] \end{array} \right) \geq \left(\begin{array}{c} [3,4] \\ [4,5] \end{array} \right) = v(\{2\}),$$

$$X^3 = \left(\begin{array}{c} [5,5] \\ [2,2] \end{array} \right) \geq \left(\begin{array}{c} [4,5] \\ [1,2] \end{array} \right) = v(\{3\}).$$

Further $X \notin C(v)$, in fact it is not valid the following:

$$X^{\{2,3\}} = X^2 + X^3 \geq v(\{2,3\})$$

Instead:

$$Y = \left(\begin{array}{ccc} [2,3] & [4,5] & [6,7] \\ [4,5] & [4,5] & [2,3] \end{array} \right) \in C(v)$$

4. Main results

Proposition 1. Let $\langle N, v \rangle$ be an interval multicriteria game. Then

- (i) $C(v) \neq \emptyset \Leftrightarrow C(v_j) \neq \emptyset, \forall j$.
- (ii) If $C(v_j) \neq \emptyset, \forall j$, then $C(\underline{v}_j) \neq \emptyset$ and $C(\bar{v}_j) \neq \emptyset, \forall j = 1, \dots, m$.
- (iii) If there is $\underline{X}_j = (a_j^1, a_j^2, \dots, a_j^n) \in C(\underline{v}_j)$ and $\bar{X}_j = (b_j^1, b_j^2, \dots, b_j^n) \in C(\bar{v}_j)$ such that $\bar{X}_j \geq \underline{X}_j \forall j$, then $([a_j^1, b_j^1], \dots, [a_j^n, b_j^n]) \in C(v_j)$.

Proof. The thesis follows from $C(v) = \prod_{j=1}^n C(v_j)$ and from results in Alparslan et al. (2009). $\square$

Example 3. To understand better the core properties in (iii) of Proposition 1 let us consider the following game $\langle N, v \rangle$:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	$[2,3]$	$[3,4]$	$[4,5]$	$[7,9]$	$[8,10]$	$[9, \frac{21}{2}]$	$[12,15]$
	$[2,3]$	$[4,5]$	$[1,2]$	$[8,10]$	$[5, \frac{13}{2}]$	$[7,9]$	$[10,13]$

We can associate to this game four games $\langle N, \underline{v}_1 \rangle$, $\langle N, \bar{v}_1 \rangle$, $\langle N, \underline{v}_2 \rangle$, $\langle N, \bar{v}_2 \rangle$ described in the following tables:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\underline{v}_1(S)$	2	3	4	7	8	9	12
S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\bar{v}_1(S)$	3	4	5	9	10	$\frac{21}{2}$	15
S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\underline{v}_2(S)$	2	4	1	8	5	7	10
S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\bar{v}_2(S)$	3	5	2	10	$\frac{13}{2}$	9	13

We can see that:

$$(3, 4, 5) \in C(\underline{v}_1),$$

$$(4, 5, 6) \in C(\bar{v}_1),$$

$$(3, 5, 2) \in C(\underline{v}_2),$$

$$(4, 6, 3) \in C(\bar{v}_2).$$

From this relation we can build an element $X \in C(v)$:

$$X = \begin{pmatrix} [3, 4] & [4, 5] & [5, 6] \\ [3, 4] & [5, 6] & [2, 3] \end{pmatrix}$$

Let $\langle N, w \rangle$ be an interval (one-criterion) game. Now we introduce the cooperative game $\langle N, |w| \rangle$ (length game) with

$$|w|(S) = \bar{w}(S) - \underline{w}(S) \geq 0, \forall S \in 2^N \setminus \{\emptyset\}.$$

So the interval game can be characterized from $\langle N, \underline{w} \rangle$, $\langle N, |w| \rangle$.

Proposition 2. *Let us suppose that the game $\langle N, |w| \rangle$ has the weak-superadditivity and the monotonic property and $|w|(\{i\}) \neq 0, \forall i \in N$. Let*

$$\frac{|w|(S)}{\sum_{i \in S} |w|(\{i\})} \leq \frac{|w|(N)}{\sum_{i \in N} |w|(\{i\})}, \forall S \in 2^N \setminus \{\emptyset\}$$

and $C(\underline{w}) \neq \emptyset$. Then $C(\bar{w}) \neq \emptyset$.

Proof. Let be $C(\underline{w}) \neq \emptyset$ and $(a_1, a_2, \dots, a_n) \in C(\underline{w})$. Let us put $b_i = a_i + \alpha(\bar{w}(\{i\}) - \underline{w}(\{i\})) \geq a_i, \forall i \in N$, where $\alpha = (|w|(N)) / (\sum_{i \in N} |w|(\{i\})) \geq 0$. Now we verify that $(b_1, b_2, \dots, b_n) \in C(\bar{w})$.

(i) Let us rewrite

$$\begin{aligned}\sum_{i \in N} b_i &= \sum_{i \in N} a_i + \alpha \sum_{i \in N} (\bar{w}(\{i\}) - \underline{w}(\{i\})) \\ &= \sum_{i \in N} a_i + \frac{|w|(N)}{\sum_{i \in N} |w|(\{i\})} \sum_{i \in N} |w|(\{i\}) = \underline{w}(N) + |w|(N) = \bar{w}(N).\end{aligned}$$

(ii) If $S \in 2^N \setminus \{\emptyset\}$ we have $\sum_{i \in S} a_i \geq \underline{w}(S)$, and

$$\alpha \sum_{i \in N} |w|(\{i\}) \geq \frac{|w|(S)}{\sum_{i \in S} |w|(\{i\})} \sum_{i \in S} |w|(\{i\}) = |w|(S) = \bar{w}(S) - \underline{w}(S) \geq \bar{w}(S) - \sum_{i \in S} a_i.$$

Otherwise if $\alpha \sum_{i \in N} |w|(\{i\}) \geq \bar{w}(S) - \sum_{i \in S} a_i$, then

$$\sum_{i \in S} b_i = \sum_{i \in S} a_i + \alpha \sum_{i \in S} (|w|(\{i\}) \geq \bar{w}(S)).$$

□

Definition 4. A multicriteria interval game is balanced (we call it a balanced game) if for each balanced map λ the following holds:

$$v(N) \geq \sum_S \lambda(S) v(S)$$

Proposition 3. The multicriteria interval game $\langle N, v \rangle$ is a balanced game if and only if $\langle N, v_j \rangle$, $j = 1, \dots, m$ are interval balanced games.

Theorem 1. Let $\langle N, v \rangle$ be a multicriteria interval game, then the following relations are equivalent:

- (i) $C(v) \neq \emptyset$
- (ii) $\langle N, v \rangle$ is a balanced multicriteria interval game.

Proof. This is a generalization of Bondareva-Shapley theorem to our games, see Owen (1995). □

Example 4. Let us consider the following game for an application of Bondareva-Shapley theorem:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	$[2,3]$	$[3,4]$	$[4,5]$	$[8, \frac{19}{2}]$	$[9,11]$	$[10, \frac{23}{2}]$	$[12,15]$
	$[2,3]$	$[4,5]$	$[1,2]$	$[9, \frac{21}{2}]$	$[6,8]$	$[6,8]$	$[10,13]$

We prove that $C(v) = \emptyset$ through the Bondareva-Shapley Theorem. A balanced map for this game is a function $\lambda : 2^N \setminus \{\emptyset\} \rightarrow [0, +\infty)$ such that:

$$\begin{aligned}\lambda(\{1\}) + \lambda(\{1,2\}) + \lambda(\{1,3\}) + \lambda(\{1,2,3\}) &= 1; \\ \lambda(\{2\}) + \lambda(\{1,2\}) + \lambda(\{2,3\}) + \lambda(\{1,2,3\}) &= 1; \\ \lambda(\{3\}) + \lambda(\{1,3\}) + \lambda(\{2,3\}) + \lambda(\{1,2,3\}) &= 1.\end{aligned}$$

The function λ satisfying $\lambda(\{1\}) = \lambda(\{2\}) = \lambda(\{3\}) = 0$ and $\lambda(\{1, 2\}) = \lambda(\{1, 3\}) = \lambda(\{2, 3\}) = \lambda(\{1, 2, 3\}) = \frac{1}{3}$ is a balanced map. But our game is not balanced, in fact the following relation is valid:

$$v(\{1, 2, 3\}) < \frac{1}{3}v(\{1, 2\}) + \frac{1}{3}v(\{1, 3\}) + \frac{1}{3}v(\{2, 3\}) + \frac{1}{3}v(\{1, 2, 3\})$$

Definition 5. Let $X, Y \in I^*(v)$, S a proper coalition, $S \subsetneq N$. We say that Y dominates X via coalition S (and we will write for short $Y \text{ dom}_S X$) if

- (i) $Y^S \geq X^S$ and
- (ii) $v(S) \geq Y^S$.

Let $D(S)$ denote the set of preimputations dominated via S . Intuitively, the payoffs distribution Y is better than $X \forall i \in S$ if (i) is valid; the payoff $(Y_i)_{i \in S}$ are attainable for the members of S thanks to cooperation if (ii) is valid. Against each $X \in D(S) = \{Z \in I^*(v) | \exists Y \in I^*(v), Y \text{ dom}_S Z\}$, the players of S can protest successfully.

A pre-imputation X is *undominated* if for any coalition $S \subset N$ there is no pre-imputation $Y \in I^*(v)$ which dominates it. We will call this set $UND(v)$.

Theorem 2. Given the interval multicriteria game $\langle N, v \rangle$, we have

$$C(v) \subseteq UND(v).$$

Proof. Let X be a preimputation s.t. $X \in C(v)$ and let us suppose by contradiction that $X \notin UND(v)$. Then there is a coalition S and $Z \in I^*(v)$, s.t. $Z \text{ dom}_S X$. Since $X \in C(v)$ and by the dominance it follows that $v(S) \geq Z^S \geq X^S \geq v(S)$ and we arrive at a contradiction. $\square$

Remark 1. The previous inclusion can be strict. Let us consider the game:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$v(S)$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[1, 3]$	$[0, 0]$	$[0, 0]$	$[1, 2]$
	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[1, 3]$	$[0, 0]$	$[0, 0]$	$[1, 2]$

We get $C(v) = \emptyset$, but $UND(v) \neq \emptyset$.

- (i) $C(v) = \emptyset$, because it is not a balanced game. It is sufficient to consider the balanced map $\lambda : 2^N \setminus \{\emptyset\} \rightarrow [0, +\infty)$ such that $\lambda(\{1\}) = \lambda(\{2\}) = \lambda(\{3\}) = \lambda(\{1, 3\}) = \lambda(\{1, 2\}) = 0$ and $\lambda(\{2\}) = \lambda(\{1, 2\}) = \lambda(\{1, 2, 3\}) = \frac{1}{2}$.
- (ii) $D(S) = \emptyset$, if $S \neq \{1, 2\}$ and $\begin{pmatrix} [0, 0] & [0, 0] & [1, 2] \\ [0, 0] & [0, 0] & [1, 2] \end{pmatrix} \in D(\{1, 2\})$.
- (iii) $I^*(v) \setminus D(\{1, 2\}) = \mathbb{C}_{I^*(v)} D(\{1, 2\}) \neq \emptyset$ (the complement of $D(\cdot)$ relative to $I^*(\cdot)$) since $Z = \begin{pmatrix} [\frac{1}{2}, 1] & [\frac{1}{2}, 1] & [0, 0] \\ [\frac{1}{2}, 1] & [\frac{1}{2}, 1] & [0, 0] \end{pmatrix} \notin D(\{1, 2\})$.
- (iv) $UND(v) \neq \emptyset$, $UND(v) = I^*(v) \setminus \bigcup_{S \in 2^N \setminus \{\emptyset, N\}} D(S) = \bigcap_{S \in 2^N \setminus \{\emptyset, N\}} \mathbb{C}_{I^*(v)} D(S) = \mathbb{C}_{I^*(v)} D\{1, 2\} \neq \emptyset$. $\square$

5. Approximate core

In the following when we write $\varepsilon' > \varepsilon$ with $\varepsilon, \varepsilon' \in \mathfrak{I}(\mathbb{R}^m)$. We mean:

$$\begin{pmatrix} [\varepsilon', \varepsilon'] \\ \vdots \\ [\varepsilon', \varepsilon'] \end{pmatrix} > \begin{pmatrix} [\varepsilon, \varepsilon] \\ \vdots \\ [\varepsilon, \varepsilon] \end{pmatrix}$$

Definition 6. Given an interval multicriteria game $\langle N, v \rangle$, $\varepsilon > 0$, let us define the ε -Core in the following way:

$$C_\varepsilon(v) = \{X \in I^*(v) : X^S + \varepsilon \geq v(S), \forall S \in 2^N \setminus \{\emptyset\}\}$$

Proposition 4. The following relations hold:

- (i) If $\varepsilon' > \varepsilon$ then $C_{\varepsilon'}(v) \supset C_\varepsilon(v)$;
- (ii) if $C(v) \neq \emptyset$, then $C(v) = \bigcap_{\varepsilon > 0} C_\varepsilon(v)$;
- (iii) there is $\bar{\varepsilon}$ minimal such that $C_{\bar{\varepsilon}}(v) \neq \emptyset$.

This particular approximate core is known as the least-core, $LC(v)$.

Proof. i) if $X \in C_\varepsilon(v)$, then $X \in C_{\varepsilon'}(v)$ because $X^S + \varepsilon' \geq X^S + \varepsilon \geq v(S) \forall S \in 2^N \setminus \{\emptyset\}$;
 ii) " $\supset$ " if $X \in C(v) = C_0(v)$, then $X \in C_\varepsilon(v)$, $\forall \varepsilon > 0$, " $\supset$ " if $X \in C_\varepsilon(v)$, $\forall \varepsilon > 0$ we have trivially the thesis. iii) There is at least $\varepsilon > 0$ such that $C_\varepsilon(v) \neq \emptyset$ so we define $LC = \bigcap_{\varepsilon > 0} C_\varepsilon(v)$ with $C_\varepsilon(v) \neq \emptyset$ and we arrive to the thesis (for more details see Owen 1995 and its references). $\square$

Example 5. Let us consider the game in the Example 4 with empty core. Let us determine the ε -Core. Let us consider $(X^1, X^2, X^3) \in C_\varepsilon(v)$. Then

$$\begin{aligned} X^1 + \varepsilon &\geq \begin{pmatrix} [2, 3] \\ [2, 3] \end{pmatrix}, X^2 + \varepsilon \geq \begin{pmatrix} [3, 4] \\ [4, 5] \end{pmatrix}, X^3 + \varepsilon \geq \begin{pmatrix} [4, 5] \\ [1, 2] \end{pmatrix}, \\ X^1 + X^2 + \varepsilon &\geq \begin{pmatrix} [8, \frac{19}{2}] \\ [9, \frac{21}{2}] \end{pmatrix}, X^1 + X^3 + \varepsilon \geq \begin{pmatrix} [9, 11] \\ [6, 8] \end{pmatrix}, \\ X^2 + X^3 + \varepsilon &\geq \begin{pmatrix} [10, \frac{23}{2}] \\ [6, 8] \end{pmatrix}, X^1 + X^2 + X^3 = \begin{pmatrix} [12, 15] \\ [10, 13] \end{pmatrix}. \end{aligned}$$

All these conditions are satisfied for $\varepsilon \geq 1$ so $LC(v) = C_1(v)$.

6. Conclusions and open problems

A first approach to multicriteria games with interval values is considered in this paper. We have generalized to this new class of cooperative games some classical concepts. Some examples have been inserted to explain the applications to real situations.

It is well known that there are other concepts of solutions for cooperative games as the Shapley value, the nucleolus, the τ -value and others (see Owen 1995; Tijs 2003). An open problem is to generalize these definitions to multicriteria games with interval values and to investigate the properties of these concepts and of the corresponding approximate concepts (also other definitions of approximate core known in literature) and to study relations among them. Another interesting approach after having studied these properties is to attempt, if possible, an axiomatic approach keeping into account the results in Miglierina et al. (2008) and Patrone et al. (1998).

At least another question arises naturally: in which ways is it possible to use this multicriteria approach for non-cooperative games? Bearing in mind the paper Patrone et al. (2007), which properties can we study about multicriteria games with interval values and potential function? Some of these topics are work in progress.

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