AUCO Czech Economic Review 3 (2009) 254–269
Acta Universitatis Carolinae Oeconomica

Ranking Electoral Systems through Hierarchical Properties Ranking

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Received 10 July 2007; Accepted 27 July 2008

Abstract Electoral systems are characterized by a wide spectrum of properties that cannot be all satisfied at the same time. We aim at examining such properties within a hierarchical framework, based on *Analytic Hierarchy Process*, performing pairwise comparisons at various levels of a hierarchy to get a global ranking of the electoral systems. In this way it should be possible to estimate the relative importance of each property with respect to the final ranking of every electoral formula.

Keywords Electoral systems, global ranking, hierarchy, aggregations
JEL classification C43, C44, D71, D72

1. The basic motivations

In this paper we aim at using a method for social rankings and at its application as a voting method (Section 5) and as a ranking tool of the properties of a set of voting methods (Sections 6 and 7) for the selection of a perfect voting system that fits at the best a set of relevant properties. The ranking method is the *Analytic Hierarchy Process* (Section 2). The method requires that each involved actor, either in isolation or in co-operation, performs the proper rankings in real cases. It can potentially avoid the theoretical hindrances of Section 4 and allow the fulfillment of the properties we list in Section 3 (Grilli di Cortona et al. 1999).

Since our primary goal was the presentation of the method and the demonstration of its use and usefulness in performing such rankings (Saaty 1980), we did not use real actors in real cases. Moreover many of the rankings have been performed having in mind the formal aspects of the *Analytic Hierarchy Process* rather than the involved properties. In this way we show the formal aspects of the method that must be tested in real cases with real actors that must perform real choices.

The paper is structured as follows. After a discussion of the mathematics of the ranking method we propose some notes on electoral systems and comment the properties we wish they satisfy. Then we present a ranking example and propose it as a voting method. The next step is the application of the ranking method to other cases so to get a certain number of orderings. The last step is the association between orderings and voting methods. The paper closes with some remarks and plans for future works.

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2. The mathematical tool

The *Analytic Hierarchy Process* (Saaty 1980; Saaty and Kearns 1985; and for applications Bhushan and Rai 2004) is a method for ranking n alternatives depending on their order of importance or preference with respect to a main goal on the basis of numerical evaluations on a ratio scale. It has been criticized in many papers, among others Bana e Costa and Vansnick (2008).

The method starts with an *analysis phase* for the identification of a set of elements and the definition of a rooted hierarchy. At the root (level $l = 0$) we have the main goal (MG), in many cases of political nature, whereas at the level of the leaves we have the alternatives.

In Figure 1 we show a complete (or fully connected between contiguous levels) hierarchy with a main goal, three actors ($ac1$, $ac2$ and $ac3$), four criteria (the cri) and three alternatives (A, B and C).

Given any level l with m elements, if we want to evaluate the importance of the n elements at level $l+1$ with respect to those at level l we build m matrices of size $n \times n$. In case of Figure 1 we have one 3×3 matrix to weigh the importance of the actors with respect to MG, three 4×4 matrices to weigh the importance of the criteria with respect to each of the actors and four 3×3 matrices to weigh the alternatives with respect to each of the criteria. This phase is carried out by the actors that, either individually or in co-operation, evaluate and properly merge the matrices of the pairwise comparisons (Saaty 1980).

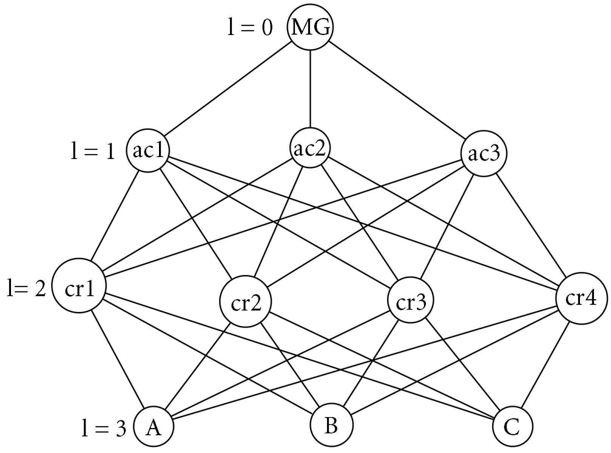

Figure 1. Example of a complete hierarchy

Each matrix A is evaluated performing pairwise comparisons between the elements of level $l+1$ with regard to those at level l. A's elements a_{ij} represent the relative importance of element i with respect to element j, assume positive values from a predefined scale and satisfy the conditions $a_{ii} = 1$, $a_{ji} = 1/a_{ij}$ and (if A is fully consistent) $a_{ij} = a_{ik}a_{kj}$ with $i, j, k = 1, \ldots, n$.

If A satisfies such properties it is called positive reciprocal and consistent. Either a_{ij} or a_{ji} only is assigned one of the following values (whereas respectively a_{ji} or a_{ij} assumes the reciprocal value, Saaty 1980): 1, 3, 5, 7, 9, respectively, to denote equal importance, weak importance, strong importance, very strong importance, absolute importance of element i over j and 2, 4, 6 or 8 as intermediate values.

Now we have the *synthesis phase* for the definition of a normalized vector of priorities of the three alternatives with respect to MG. This calculation turns into a series of eigenvalue/eigenvector problems whose full treatment (but for few details) is out of the scope of this paper (see Saaty 1980).

When each matrix A has been evaluated we can associate to it the normalized vector of the weights $\mathbf{w} = (w_1, \ldots, w_n)$ with $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$ such that $A\mathbf{w} = n\mathbf{w}$. $\mathbf{w}$ is the eigenvector of A and $\lambda_{max} = n$ is the associated main eigenvalue of A (if it is fully consistent), all the others being equal to 0.

If A is not fully consistent, $\lambda_{max} \approx n$. In this case the normalized eigenvector $\mathbf{w}'$ represents a proxy of the real eigenvector $\mathbf{w}$ that is the better the more λ_{max} tends to n. The method has a criterion that allows the evaluation of the consistency of the matrix A. If A is fully consistent we get n identical values, otherwise we get n slightly different values that we can average to get the true value of λ_{max} to be used to evaluate the degree of consistency of the matrix.

The criterion uses the *consistency index* $= (\lambda_{max} - n)/(n-1)$ that is compared with the average *random index* (provided in Saaty 1980 for different values of n) that represents the consistency index of a randomly generated reciprocal matrix on the same scale and allows us to obtain the *consistency ratio* $=$ *consistency index*/*random index*.

Values of *consistency ratio* equal to 0.0 define a fully consistent matrix, lower than 0.10 define a consistent matrix, values between 0.10 and 0.20 must be considered with care, values greater than 0.20 (Saaty and Kearns 1985, p. 34) should impose a revision of the judgments. The problem of consistency arises only when $n > 2$.

We have now the matrices A_i, each with the eigenvalue λ_i and eigenvector $\mathbf{w}_i$. If A_1 is the matrix of the n_1 elements at level 1 with respect to MG (level 0), we have a vector $\mathbf{w}_1 = L_1$ of the weights. If at level 2 we have n_2 elements, we get n_1 matrices of size $n_2 \times n_2$ and therefore n_1 eigenvectors $\mathbf{w}_i$ of n_2 elements each. We can construct an $n_2 \times n_1$ matrix L_2. If we want to evaluate the weights of the elements at level 2 with respect to MG, we can simply evaluate the product[1] L_2L_1 or a normalized vector of n_2 elements. We can define the matrices of the pairwise comparisons of the elements at level 3 with respect to those at level 2, be it A_2, and define the matrix L_3 of the vectors of the weights. In order to get the weights of the elements at level 3 with respect to MG we can evaluate $L_3L_2L_1$ or a normalized vector of n_3 elements.

The *last step* is a set of procedures for the evaluation of the normalized eigenvectors from the matrices A_i without solving their characteristic equations. For our computations (Saaty 1980) we used the method of the n-th root of the product. To apply it we

[1] We use row vectors and no symbol to denote transposition. Depending on the context a row vector must be seen as column vector.

multiplied the elements of each row among themselves, evaluated the *n*-th root (if *n* is the dimension of the matrix) of that value and, lastly, normalized the resulting vector.

3. The desired properties or the wish lists

We start with a first *wish list* of basic properties that are involved in Arrow's Theorem. (i) *Universal Domain* implies that the chosen aggregation method must be universally applicable so that from any rankings provided by the voters it must yield an overall ranking of the candidates so to rule out "methods that would impose some restrictions on the preferences of the voters" (Bouyssou et al. 2000, p. 17). (ii) *Transitivity* requires that the aggregation of the rankings must be a ranking that satisfies transitivity. (iii) *Unanimity* or *Pareto Condition* implies that, if each voter ranks a candidate higher than another, this ranking must be reflected in the overall ranking. (iv) *Binary Independence* (or Independence from Irrelevant Alternatives) requires that the relative position of two candidates in the overall ranking depends only on their relative position in each voter's ranking so that all the other alternatives are seen as irrelevant. (v) *Non-dictatorship* means that there is no voter that can impose his ranking as the overall social ranking.

The Condorcet method satisfies properties (i), (iii), (iv) and (v) so that, by Arrow's Theorem, it must fail property (ii) whereas the Borda method satisfies properties (i), (ii), (iii) and (v) so that, by Arrow's Theorem, it must fail property (iv).

We can add the following properties, that may take a different meaning for proportional and majoritarian methods. (vi) *Anonymity* (Taylor 2005) requires that the overall ranking is independent from any permutation of the voters. (vii) *Neutrality* (Taylor 2005) means that the overall ranking is independent from any permutation of the alternatives. (viii) *Separability* (Bouyssou et al. 2000) requires that if we perform an election with two separate sets of voters and obtain a winner candidate on each set such candidate remains a winner if we repeat the election with the same method on the union of the two sets of voters. (ix) *Monotonicity* (Bouyssou et al. 2000, p. 11) requires that "an improvement of a candidate's position in some of the voter's preferences cannot lead to a deterioration of his position after the aggregation". (x) *Non-manipulability* essentially means that the overall ranking of a set of candidates does not depend either on the agenda or on the presence of straw candidates or on the expression of non true preferences.

Majoritarian methods are characterized by the following properties. *Condorcet Winner* (CW) is the winner of all pairwise comparisons, if it exists it should be the winner of the electoral competition. *Condorcet Loser* (CL): a method should not choose the candidate that loses every pairwise comparison with all the other candidates. *Monotonicity* (M): a method is monotone if the number of seats assigned to a party does not decrease if the number of its supporters grows. *Pareto Principle* (PP): if all the voters prefer a candidate to another the latter cannot be chosen. *Weak Axiom of Revealed Preference* (WARP): it requires that, (a), if a candidate is a winner on a set X of candidates it must remain a winner also on any subset $X' \subseteq X$ to which he belongs and that, (b), if there are ties among candidates in $X' \subseteq X$ those candidates at par must be all either included or excluded from the final set of winners in X. This axiom is used to get voting methods immune from manipulations on the set of candidates through the addition of straw candidates. *Path Independence* (PI): a method satisfies path independence if the outcome is independent from the ordering of the phases that are used for the selection of the candidates.

We note that *Plurality* method satisfies (vi), PP and WARP whereas *Double Ballot* and *Single Transferable Vote*[2] methods satisfy (vi), CL and PP.

Proportional methods are characterized by the following properties. *House Monotonicity* (HM) means that if the number of seats passes from S to $S+1$ no party gets fewer seats. *Quota Satisfaction* (QS) requires that the number of seats each party receives is as close as possible to its exact quota and so to a percentage of the total seats that is almost equal to the percentage of the votes it receives. *Population Monotonicity* (PM) (Grilli di Cortona et al. 1999) "a party (or state) with a growing weight cannot lose a seat in favor of a party (or state) with a declining weight". *Consistency* (C) requires that any partial assignment is itself proportional. *Stability* (S) means that whenever two parties merge in a coalition (or a new party) they do not get fewer seats that those they get as separate entities.

We note that the *Quota* method satisfies (vi), HM, QS, C (only with regard to pairs of eligible parties) and (v), *Divisor* methods satisfy (vi), HM, PM, C and S (only in particular cases) whereas *Largest remainders* methods satisfy (vi), QS and S.

4. Some impossibility results

An electoral system represents a very complex process that can be decomposed in a certain number of phases and whose performance can be measured with a set of criteria and indicators (Grilli di Cortona et al. 1999).

An electoral system, starting from each voter's ranking of a set of alternatives (the candidates) from the best to the worse, aims at aggregating such rankings in a global social ranking. This is a very hard task and literature is full of impossibility results. The most famous is Arrow's impossibility Theorem (Bouyssou et al. 2000): with more than two candidates there is no aggregation method that can satisfy the properties of Universal Domain, Transitivity, Unanimity or Pareto condition/principle, Binary Independence and Non-dictatorship.

Another result is Sen's Theorem (Saari 2001), based on a condition of Minimal Liberalism (ML).[3] It states that with more than two alternatives and two or more voters, if Universal Domain, ML and Pareto are satisfied we are bound to have profiles (or sets of preferences) that have cyclic outcomes. We mention also Gibbard-Satterthwaite's Theorem (Bouyssou et al. 2000) that concerns strategic voting (or the convenience of not expressing one's true preferences) and that states that with more than two candi-

[2] In Grilli di Cortona et al. (1999, p. 29), *Single transferable vote* is cited among proportional methods whereas in the Table at p. 78 it is put in comparison with other "most popular majoritarian methods". We chose the latter classification for our comparisons of majoritarian methods, section 6.

[3] A Social Welfare Function is is said to satisfy ML if (Saari 2001) each of at least two voters is decisive over a pair of alternatives so that his ranking of such pair determines that pair's societal ranking.

dates there exists no aggregation method that satisfies simultaneously the properties of Universal Domain, Non-manipulability and Non-dictatorship.

Other results may be found in Balinski and Peyton Young (1982), as the impossibility of satisfying both population monotonicity and staying within for proportional representation methods.

The properties we have listed with Arrow's impossibility theorem are really minimal for any real democratic process and things are even worse (Bouyssou et al. 2000) with additional properties such as Neutrality, Separability, Monotonicity, Non-manipulability and so on. Similar considerations hold also for Gibbard-Satterthwaite and Sen's Theorems.

5. A ranking of alternatives

We have three voters vi ($i = 1,2,3$) that rank four alternatives aj ($j = 1,2,3,4$, e.g. candidates in an electoral competition) so to define a total ordering with possible ties (see Figure 2) .

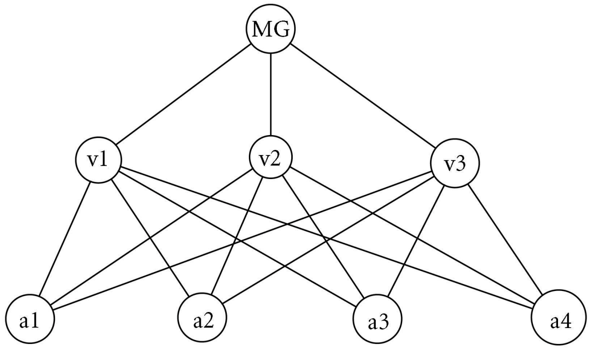

Figure 2. Three voters and four alternatives

We wish to evaluate the normalized vector **w** of the weights of the voters with regard to MG. Imposing a full symmetry we get a fully consistent 3×3 matrix (with all elements equal to 1) to which it corresponds the eigenvalue $\lambda_{max} = 3$ and a normalized eigenvector $L_1 = (1/3, 1/3, 1/3)$. This result is consistent with our intuition of a fair evaluation tool where the three voters have the same weight. Then we evaluate one 4×4 matrix of the pairwise comparisons of the four alternatives for each voter according to the following preferences ($>$ denotes strict preference and $\sim$ indifference): $v1$ has $a1 > a2 > a3 > a4$, $v2$ has $a1 > a2 > a3 > a4$ and $v3$ has $a3 \sim a4 > a2 \sim a1$. The three matrices (Table 1) satisfy at the best the requirements of *Analytic Hierarchy Process* and reflect the voters' judgments.

Table 1. Pairwise comparisons with regard to $v1$, $v2$ and $v3$ (from left to right)

v1	a1	a2	a3	a4	v2	a1	a2	a3	a4	v3	a1	a2	a3	a4
a1	1	2	5	7	a1	1	2	1/2	1/4	a1	1	1	1/5	1/5
a2	1/2	1	2	3	a2	1/2	1	1/3	1/6	a2	1	1	1/5	1/5
a3	1/5	1/2	1	2	a3	2	3	1	1/3	a3	5	5	1	1
a4	1/7	1/3	1/2	1	a4	4	6	3	1	a4	5	5	1	1

Note: We use fractions to underline the relation between a_{ij} and a_{ji}.

We evaluate the eigenvectors (see Table 2), the corresponding eigenvalues and verify that each matrix is consistent.

Table 2. Matrix L_2 of the eigenvectors alternatives versus voters

	$\mathbf{w}_1$	$\mathbf{w}_2$	$\mathbf{w}_3$
$L_2 =$	0.5488	0.1355	0.0833
	0.2497	0.0782	0.0833
	0.1269	0.2279	0.4167
	0.0745	0.5583	0.4167

The vector $\mathbf{w}$ is $\mathbf{w} = L_2L_1 = (0.2559, 0.1371, 0.2571, 0.3498)$ and gives the $a4 > a3 > a1 > a2$ ordering on the alternatives.

At this point we have to understand what we got and for what. We got a ranking but can we use it as if it was an election outcome? Maybe. The main problem is the inconsistency issue. In the general case, indeed, we can have one or more inconsistent matrices. How can we deal with this? There is any threshold above which we should repeat a ranking? Or should we consider it anyway valid? Some of these questions will remain unanswered, others will find partial answers in Section 8.

6. Some rankings of properties

We deal with abstract properties of the families of majoritarian and proportional methods to obtain a ranking of those properties so to define the perfect method within each family.

We start with four voters who rank the six main properties of proportional methods (see Figure 3): (vi), *Anonymity* (A), *House Monotonicity* (HM), *Quota Satisfaction* (QS), *Population Monotonicity* (PM), *Consistency* (C) and *Stability* (S). The voters have the following preference orderings: $v1$ has $A > HM > QS > PM > C > S$, $v2$ has $A \sim HM > QS \sim PM > C > S$, $v3$ has $S > C \sim PM > A > HM \sim QS$ and $v4$ has $QS > HM \sim A > PM \sim C \sim S$.

If each voter performs the pairwise rankings we get the Tables 3 and 4.

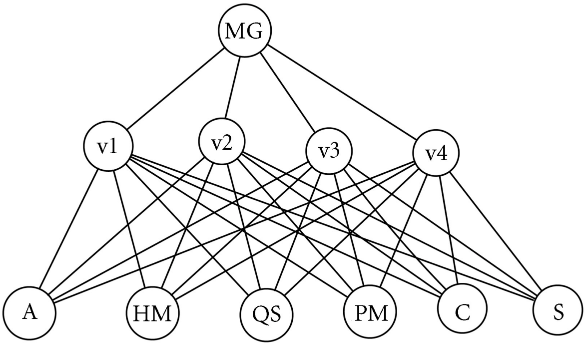

Figure 3. Ranking properties of proportional methods

Table 3. Pairwise comparisons with regard to *v*1 and *v*2

v1	A	HM	QS	PM	C	S	v2	A	HM	QS	PM	C	S
A	1.00	2.00	3.00	4.00	6.00	9.00	A	1.00	1.00	3.00	3.00	5.00	7.00
HM	0.50	1.00	2.00	2.00	3.00	4.00	HM	1.00	1.00	3.00	3.00	5.00	7.00
QS	0.33	0.50	1.00	1.00	2.00	2.00	QS	0.33	0.33	1.00	1.00	5.00	7.00
PM	0.25	0.50	1.00	1.00	2.00	2.00	PM	0.33	0.33	1.00	1.00	5.00	7.00
C	0.17	0.33	0.50	0.50	1.00	2.00	C	0.20	0.20	0.20	0.20	1.00	2.00
S	0.11	0.25	0.50	0.50	0.50	1.00	S	0.14	0.14	0.14	0.14	0.50	1.00

Table 4. Pairwise comparisons with regard to *v*3 and *v*4

v3	A	HM	QS	PM	C	S	v4	A	HM	QS	PM	C	S
A	1.00	2.00	2.00	1.00	1.00	0.20	A	1.00	1.00	0.33	2.00	2.00	2.00
HM	0.50	1.00	1.00	0.50	0.50	0.14	HM	1.00	1.00	0.33	2.00	2.00	2.00
QS	0.50	1.00	1.00	0.50	0.50	0.14	QS	3.00	3.00	1.00	7.00	7.00	7.00
PM	1.00	2.00	2.00	1.00	1.00	0.33	PM	0.50	0.50	0.14	1.00	1.00	1.00
C	1.00	2.00	2.00	1.00	1.00	0.33	C	0.50	0.50	0.14	1.00	1.00	1.00
S	5.00	7.00	7.00	3.00	3.00	1.00	S	0.50	0.50	0.14	1.00	1.00	1.00

We evaluate the eigenvectors (the leftmost four columns of Table 5), the corresponding eigenvalues and verify that each matrix is consistent. Since the four voters have the same weight with regard to MG $L_1 = (0.25, 0.25, 0.25,\ 0.25)$ so that as $\mathbf{w} = L_2 L_1$ we get the fifth column of Table 5 where the sixth column contains the listing of the mnemonics of the properties and the last their place in the classification.

A comparison of the classification with the results of the table at page 83 of Grilli di Cortona et. al (1999) allow us to assert that the best proportional method is the *Quota* method. From that table we have that *Quota* method satisfies A, HM, QS, C (but only in special cases) and S, and that $\mathbf{w}_A + \mathbf{w}_{HM} + \mathbf{w}_{QS} = 0.65$.

Table 5. Eigenvectors alternatives versus voters and final ranking of the alternatives

	$\mathbf{w}_1$	$\mathbf{w}_2$	$\mathbf{w}_3$	$\mathbf{w}_4$	$\mathbf{w}$	
A	0.43	0.31	0.13	0.15	0.25	1
HM	0.22	0.31	0.07	0.15	0.19	3
QS	0.12	0.15	0.07	0.48	0.21	2
PM	0.11	0.15	0.14	0.07	0.12	5
C	0.07	0.05	0.14	0.07	0.08	6
S	0.05	0.03	0.47	0.07	0.16	4

Let us suppose that $v2$ changes his preference ordering as $PM > A \sim C > S > HM > QS$. If we evaluate the new eigenvector $\mathbf{w}_2$ we get a different second column of the Table 5 and a different final priority vector $\mathbf{w} = L_2 L_1 = (0.23, 0.13, 0.18, 0.12, 0.17, 0.18)$. In this way we have that the ordering of the properties is $A \succ QS \sim SPM \succ HM \succ C$. From a comparison of that ordering with the results of the table at page 83 of Grilli di Cortona et. al (1999) we get that the best proportional method is the *Largest remainder* methods. From that table we have that *Largest remainders* methods satisfy A, QS and S; *Divisor* methods satisfy A, HM, PM, C and S (but only in special cases) and *Quota* method satisfies A, HM, QS, C (but only in special cases) and S.

The properties A, QS and S count for almost 60% over the total of the six properties and a change of one voter's opinion over four can be seen as a change of the opinion of the 25% of the voters.

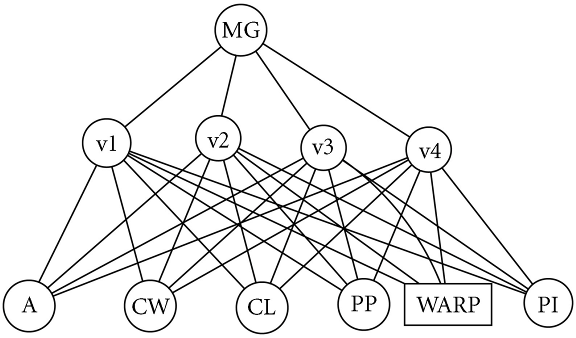

Figure 4. Ranking properties of majoritarian methods

We have a similar example with the properties of majoritarian methods. In Figure 4 we suppose to have four voters that rank the six main properties of these methods: (vi), *Anonymity* (A), *Condorcet Winner* (CW), *Condorcet Loser* (CL), *Pareto Principle* (PP), *Weak Axiom of Revealed Preferences* (WARP) and *Path Independence* (PI).

We give only the matrix L_2 of the eigenvectors and the vector of the priorities of the properties with regard to MG. The four matrices that give the eigenvectors of L_2 are based on the following preference orderings: $v1$ has $A > CW > CL > PP > WARP > PI$, $v2$ has $CW \sim CL > PP > PI > A \sim WARP$, $v3$ has $PP > A > PI > WARP > CW \sim CL$ and $v4$ has $PP > PI > A \sim WARP > CW > CL$.

Table 6. Majoritarian methods: the eigenvectors of the weights $\mathbf{w}_i$ and the final vector of the weights $\mathbf{w}$

	$\mathbf{w}_1$	$\mathbf{w}_2$	$\mathbf{w}_3$	$\mathbf{w}_4$	$\mathbf{w}_0$	$\mathbf{w}$
A	0.49	0.05	0.21	0.17	0.25	**0.23**
CW	0.16	0.35	0.06	0.05	0.25	0.15
CL	0.13	0.35	0.06	0.03	0.25	0.14
PP	0.11	0.12	0.43	0.42	0.25	**0.27**
WARP	0.07	0.05	0.10	0.08		0.07
PI	0.04	0.08	0.15	0.24		0.13

Note: The first four columns are the columns of the matrix L_2 of the eigenvectors of the alternatives as to the voters. The fifth column contains the eigenvector L_1 of the weights of the voters as to MG whereas the sixth column represents the vectors of the weights of the alternatives as to MG.

Such matrices are consistent and the final results are those of Table 6. The sixth column is obtained by a matrix vector multiplication between the first four columns and the fifth column. From these values we can devise the ordering $PP > A > CW > CL > PI > WARP$ with $\mathbf{w}_{PP} + \mathbf{w}_A + \mathbf{w}_{CL} = 0.64$ and $\mathbf{w}_{PP} + \mathbf{w}_A + \mathbf{w}_{WARP} = 0.57$.

Such results, in the light of the table at page 78 of Grilli di Cortona et. al (1999), can be a little bit difficult to interpret. From that table we have that: *Plurality* method satisfies A, PP and WARP; *Double Ballot* and *Single Transferable Vote* methods satisfy A, CL and PP; *Approval Voting* method satisfies A, WARP and PI. By confronting such information we can say that both *Double ballot* and *Single transferable vote* methods satisfy only A, CL and PP but only *Plurality* method satisfies A, PP and WARP. We can therefore devise the preference ordering *Single Transferable Vote* $\sim$ *Double Ballot* $>$ *Plurality* and reach a final decision by using other criteria.

7. Two more examples of ranking

We show two more examples of ranking electoral systems. In the first example we consider properties such as (i) *Universal Domain* (UD), (ii) *Transitivity* (TR), (iii) *Pareto*

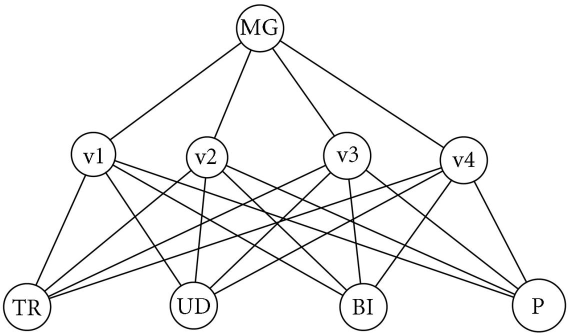

Figure 5. Ranking electoral systems through ranking some basic properties

Table 7. A first ranking of electoral systems, case of $v1$ and $v2$

v1	TR	UD	BI	P	v2	TR	UD	BI	P
TR	1.00	3.00	3.00	3.00	TR	1.00	1.00	2.00	0.33
UD	0.33	1.00	1.00	1.00	UD	1.00	1.00	2.00	0.20
BI	0.33	1.00	1.00	1.00	BI	0.50	0.50	1.00	0.14
P	0.33	1.00	1.00	1.00	P	3.00	5.00	7.00	1.00

Table 8. A first ranking of electoral systems, case of $v3$ and $v4$

v3	TR	UD	BI	P	v4	TR	UD	BI	P
TR	1.00	3.00	5.00	3.00	TR	1.00	0.17	3.00	0.25
UD	0.33	1.00	2.00	1.00	UD	6.00	1.00	9.00	2.00
BI	0.20	0.50	1.00	0.50	BI	0.33	0.11	1.00	0.14
P	0.33	1.00	2.00	1.00	P	4.00	0.50	7.00	1.00

Condition (P) and (iv) *Binary Independence* (BI) and four voters that perform a ranking of these properties (see Figure 5).

The four matrices of the pairwise comparisons (Tables 7 and 8) are based on the following preference orderings of the voters: $v1$ has $TR > UD \sim BI \sim P$, $v2$ has $P > TR > UD > BI$, $v3$ has $TR > P \sim UD > BI$ and $v4$ has $UD > P > TR > BI$.

All the matrices are consistent and the normalized eigenvectors are those of the first four columns of Table 9 where the fifth column represents the eigenvector of the matrix of the pairwise comparisons of the four voters with regard to MG. From both calculations and fairness considerations such a vector has all components equal to 0.25. The sixth column gives the global weights or priorities of the four properties with regard to MG. We have $TR \sim P > UD > BI$. Such a ranking is satisfied, for instance, by the *Borda count* (that does not satisfy binary independence) that therefore can be legitimately chosen.

We note indeed that $\mathbf{w}_{TR} + \mathbf{w}_{UD} + \mathbf{w}_P = 0.90$ so that *Binary independence* can be surely neglected.

Table 9. The eigenvectors of the weights $\mathbf{w}_i$ and the final vector of the weights $\mathbf{w}$

	$\mathbf{w}_1$	$\mathbf{w}_2$	$\mathbf{w}_3$	$\mathbf{w}_4$	$\mathbf{w}_0$	$\mathbf{w}$
TR	0.50	0.17	0.53	0.10	0.25	**0.32**
UD	0.17	0.15	0.19	0.54	0.25	**0.26**
BI	0.17	0.08	0.10	0.04	0.25	0.10
P	0.17	0.60	0.19	0.32	0.25	**0.32**

Note: The first four columns are the columns of the matrix L_2 of the eigenvectors of the alternatives as to the voters. The fifth column contains the eigenvector L_1 of the weights of the voters as to MG whereas the sixth column represents the vectors of the weights of the alternatives as to MG.

The other example involves a ranking of majoritarian M and proportional methods P (Figure 6). We have three voters ($v1$, $v2$ and $v3$) that use four properties ($p1$, $p2$, $p3$ and $p4$) to obtain a ranking between M and P to see which is better.

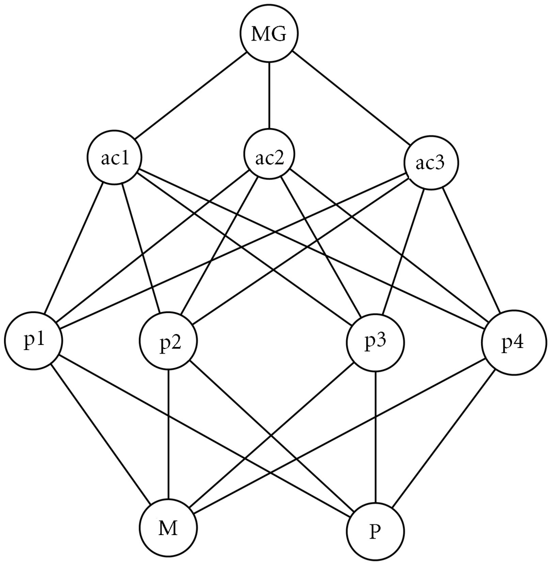

Figure 6. Majoritarian or proportional? The basic dilemma

In Figure 6 we have a rooted hierarchy where the leaves are at level 3 so we have to define the matrix L_3, the matrix L_2 and the vector L_1 and evaluate the vector $\mathbf{w}$ of the two alternatives M and P with respect to MG as $\mathbf{w} = L_3 L_2 L_1$.

From considerations we have already made $L_1 = (0.33, 0.33, 0.33)$. The hard part is the definition of the four properties. We can try with the following set (Grilli di Cortona et. al 1999): (xi) *Electoral Participation* (EP) defined as the ratio between the number of vote cast and the difference between the total number of voters and the number of vote cast; (xii) *Number of Political Parties* (NPP) defined through parameters that count both the number of parties that compete in a given election and their relative strength; (xiii) *Electoral Volatility* (EV) as a measure of the electoral fluxes among the competing parties from one electoral competition to the successive one; (xiv) *Government Stability* (GS) measured as a function of the longevity of the governments.

The actors are supposed to act according to the following preference orderings: $v1$ has $EP > NPP > GS \sim EV$, $v2$ has $EP > EV > NPP \sim GS$ and $v3$ has $GS > NPP > EP > EV$.

The four matrices at level 3 (Table 10) are the outcome of a collaborative process involving the voters through which they rank the two alternatives (M and P) according to the four properties EP, NPP, EV and GS.

Individually they rank those properties as to each one's system of values (Table 11) as represented by their preference orderings.

Table 10. Matrices of the pairwise comparisons with regard to the properties

EP	M	P	NPP	M	P	EV	M	P	GS	M	P
M	1.00	0.33	M	1.00	0.50	M	1.00	0.20	M	1.00	4.00
P	3.00	1.00	P	2.00	1.00	P	5.00	1.0	P	0.25	1.00

Note: Every group of three columns is a matrix of the pairwise comparisons of the alternatives with regard to each property.

Table 11. Pairwise comparisons with regard to $v1$, $v2$ and $v3$ form left to righ

v1	EP	NPP	EV	GS	v2	EP	NPP	EV	GS	v3	EP	NPP	EV	GS
EP	1.0	5.0	7.0	7.0	EP	1.0	7.0	3.0	7.0	EP	1.0	0.5	2.0	0.2
NPP	0.2	1.0	2.0	2.0	NPP	0.14	1.0	0.5	1.0	NPP	2.0	1.0	3.0	0.33
EV	0.14	0.5	1.0	1.0	EV	0.33	2.0	1.0	2.0	EV	0.50	0.33	1.0	0.11
GS	0.14	0.5	1.0	1.0	GS	0.14	1.0	0.5	1.0	GS	5.0	3.0	9.0	1.0

It is easy to see how the matrices of Table 10 are fully consistent whereas the matrices of Table 11 are consistent.

At this point we have a 2×4 matrix L_3 of the eigenvectors of the priorities of the alternatives with respect to the properties, a 4×3 matrix L_2 of the eigenvectors of the priorities of the properties with respect to the actors and a vector $L_1 = (0.333, 0.333, 0.333)$ that is the eigenvector of the priorities of the actors with respect to MG.

We can obtain the priorities of the two alternatives with respect to MG as $\mathbf{w} = L_3L_2L_1 = (0.3970, 0.6029)$ so to get $P > M$ and say that proportional methods are preferred to majoritarian methods. The next step would be the choice, through an analogous procedure, of one of the many available proportional methods.

8. Open theoretical problems

In the previous sections we introduced *Analytic Hierarchy Process* as either a voting system or as a tool for the ranking of electoral systems.

In the former case (but similar considerations hold also in the latter case) we have a hierarchy where three voters rank four alternatives so to define *a social choice function* of such alternatives. Are we sure in this way we got an electoral system that proves to be immune from the *contagion* of Arrow's Theorem and the other results we listed in Section 4? Saaty (1980, p. 52–53) and Saaty and Kearns (1985, p. 198–199) are confident this is the case. Also Saari (2001), a more neutral source, is quite sure that this is the case.

Saari (2001) shows how to overcome such theoretical limitations by using methods that do not miss meaningful information in the execution of pairwise comparisons between candidates or alternatives. In presence of more than one level of aggregation (see the left side of Figure 7), there can be discrepancies on the rankings (according to common criteria) of the alternatives (the leaves of the binary tree) among the various levels of inner nodes.

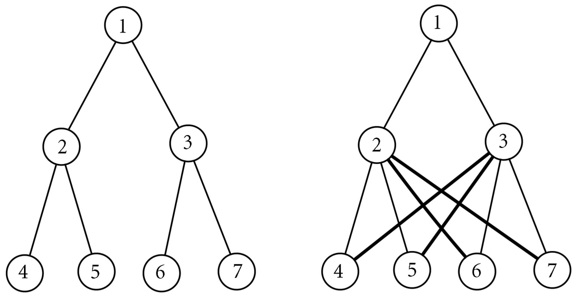

Figure 7. Binary [rooted] tree versus [rooted] hierarchy: meaningful information (in bold)

The problem occurs since nodes 2 and 3 do not share the same information. The solution that we (after Saaty and Saari) propose is the use of a complete rooted hierarchy where missing meaningful information is recovered with the bold face arcs (see the right side of Figure 7). Through a complete hierarchy we can execute pairwise comparisons of all elements at level $i+1$ among themselves for any element at level i and compose the results up to the root of the hierarchy. The proposed method is therefore a potential theoretical solution to the problem of defining a perfect voting system.

Many problems are yet present and beg for a solution. The first problem is how the system we showed in Section 5 can scale to be used as a voting system when many more voters and alternatives are present.

An increase in the number of alternatives makes the ranking problem harder so that the probability of producing inconsistent matrices becomes higher and higher with that number. Since it is usually impossible to reduce the number of alternatives at a manageable level one solution is the use of clustering techniques together with the use of hierarchies with more levels that those used in Section 5.

As to the number of voters we note how in many cases it is fixed by political rules so that it must be seen as a parameter of our method on which we cannot act but indirectly. Since, however, the profiles of ranking over a fixed scale tend to be repeated one possible solution is to gather common profiles as prototype voters and to assign each of them a weight that is a measure of the number of voters that have that profile. In this way we can reduce the number of voters to manageable quantities.

Therefore we need to evaluate to what extent the criticism moved to Saaty's method in Bana e Costa and Vansnick (2008) (see Section 2) may weaken our application, how experts or actors can rank alternatives with regard to properties or policies (see Figure 6), how to take into account the point of views and the goals of voters and candidates and, lastly, how to frame our approach among the other proposed approaches (Grilli di Cortona et al. 1999) so to put in evidence its strengths and weaknesses.

Another open problem is that of understanding how actors can evaluate the alternatives with respect to the properties (see Figure 6) and if any solution can work also for many more actors and alternatives.

The ranking from experts or actors involves the attainment of a consensus among them either as a co-ordinate and co-operative simultaneous effort or as a two step process where (a) each of them produces all the pairwise rankings, including those of the others, and (b) such rankings are merged (through an averaging of some sort) in the appropriate global rankings.

One more problem we mention is that of inconsistencies since we have to understand if we have to care of any inconsistency, in which way and if there is any inconsistency threshold (beyond the value of 0.10) above which we should declare any voting outcome as null.

Last but not least there is the possibility that a more subtle and perverse version of Arrow's Theorem is lurking out there. In this case *Analytic Hierarchy Process* would prove nothing more than another blind alley (at least for the search of a perfect voting system). A solution to this yet open problem can derive only from further theoretical and empirical investigations both within the framework of *Analytic Hierarchy Process* and within the area of electoral systems.

9. Concluding remarks and suggestions for future research

We presented an original approach for both the ranking of electoral systems and the definition of a voting method. This approach is based on a complete rooted hierarchy. At the root we have the *MG* whereas at the leaves we put the objects we want to rank through the hierarchy. The paper represents a starting point, much more work needs indeed to be done both from a theoretical and from an empirical point of view.

As to the theoretical aspects it should be interesting to investigate both the properties of the proposed voting method (Section 5) and the properties of the methods for the ranking of electoral systems (Sections 6 and 7) to see if they can be used for the selection of an electoral system among the many that can be conceived.

From the empirical point of view future research includes the testing of the proposed approach with some experiments with the involvement of both students in social sciences, experts in voting systems and simple citizens voters.

Acknowledgment The author wish to thank Professor Franco Vito Fragnelli for his help, Professor Giorgio Gallo for friendly advice and corrections and the anonymous referees that helped me in improving the quality of the paper. All errors of any kind are author's sole responsibility.

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