AUCO Czech Economic Review 2 (2008) 21–27
Acta Universitatis Carolinae Oeconomica

Partial Cooperation and Non-Signatories Multiple Decision

Lina Mallozzi[*], **Stef Tijs**[†]

Received 19 November 2007; Accepted 10 January 2008

Abstract In this paper we investigate partial cooperation between a portion of the players and the rest of the players who do not cooperate and play a Nash game having multiple equilibria. Some properties of the partial cooperative equilibrium are studied and applied to a public goods situation.

Keywords noncooperative games, cooperation, public goods games
JEL classification C71, C72, H41

1. Introduction

In the last decades the problem of international pollution control has been approached in a Game Theory setting, cooperative as well as non-cooperative. Particularly, several papers have been devoted to the coalition formation process and the stability of the formed coalition (Ray and Vohra 1997, Yi 1997, Finus 2001 and the references therein).

In the context of International Environmental Agreements (IEA), in a competition between several countries, usually only a portion of the participants signs an agreement. The IEA framework, together with other situations, leads to study concepts of partial cooperation, a mixture of cooperative behavior and non-cooperative one. It is supposed that the non-coalition members choose their strategy according to Nash behavior and the coalition maximize the aggregate welfare of its members.

The definition of partial cooperative equilibrium has been given by Mallozzi and Tijs (2006) for symmetric potential games together with existence results. In line with the definition used by Barrett (1994), it has been supposed that the cooperating players choose the same strategy and the non-cooperating ones react by playing a Nash equilibrium problem admitting a unique solution. The case where the non-cooperating reaction set is not a singleton has been studied in Mallozzi and Tijs (2007) for symmetric aggregative games.

In this paper we present the notion of partial cooperative equilibrium in the non symmetric case and an existence result. Then an application to a public goods game is

[*] University of Naples Federico II, Department of Mathematics and Applications, Via Claudio 21, 80125 Naples, Italy. Phone +390817682476, E-mail: mallozzi@unina.it.
[†] University of Tilburg, Center and Department of Econometrics and Operations Research, P.O.Box 90153, 5000 LE Tilburg, Netherlands. Phone +31134662348, E-mail: S.H.Tijs@uvt.nl.

discussed. The paper is organized as followed: in Section 2 a more general definition of partial cooperative equilibrium is given and some results are proved; in Section 3 the results are applied to a public goods game; Section 4 concludes indicating some possible generalizations of the obtained results.

2. λ-partial cooperation

Let $\Gamma = \langle n; X; f_1, ..., f_n \rangle$ be an n-person normal form game with player set $I = \{1, 2, ..., n\}$, with the same strategy space X and payoff function $f_i \colon X^n \mapsto \mathscr{R}$ for player $i \in I$. If player i chooses $x_i \in X$, then he obtains a profit $f_i(x_1, ..., x_n)$. Each player wants to maximize his own profit. We denote by x_{-i} the vector $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in X^{n-1}$.

A noncooperative behavior between the n players is described by the well known concept of Nash equilibrium. A Nash equilibrium is a vector $\eta = (\eta_1, ..., \eta_n) \in X^n$ such that for any $i \in I$

$$f_i(\eta_1, \ldots, \eta_n) = \max_{y \in X} f_i(\eta_1, \ldots, \eta_{i-1}, y, \eta_{i+1}, \ldots, \eta_n).$$

Let us denote by NE the set of the Nash equilibrium profiles of the game Γ. For any $j = 1, ..., n$ we denote by $y_{\mathbf{j}}$ the j-dimensional vector $(y_1, ..., y_n)$.

We suppose now that a group of the n players participate in an agreement, say $P_{k+1}, ..., P_n$ (cooperating players or signatories), the remaining players $P_1, ..., P_k$ (non-cooperating players or non-signatories) acting in a noncooperative way for each $k = 0, ..., n$. In this case k is called the level of *non*-cooperation. The game is a two-stage game: signatories behave as a Stackelberg leader[1] and announce their joint strategy. Non-signatories are the followers and react by playing a non-cooperative game: they choose a Nash equilibrium in the k-person subgame. The solution is given then by using backward induction. For $k = 0$ all the players are signatories and maximize their joint payoff $\sum_j f_j(y_{\mathbf{n}})$; for $k = n$ all the players are non-signatories and we have a Nash game for all n players.

More precisely, given the level of *non*-cooperation k, the signatories choose the same strategy $x_{k+1} = x_{k+2} = ... = x_n = y \in X$ and the first k players with payoffs $f_i(x_1, ..., x_k, y_{\mathbf{n-k}})$ for any $i = 1, ..., k$ do not participate in the agreement and choose a Nash equilibrium against the joint strategy $y \in X$. This uniform strategy choice of signatories may appear restrictive, but it is common in IEA models where it means the available level (for example, the percentage) of a certain gas emission. A possibility to avoid this assumption is discussed in Section 4.

Denote by $\Gamma_k(y) = \langle k; X; f_1, ..., f_k \rangle$ the k-person game with strategy space X and payoff function $f_i(x_1, ..., x_k, y_{\mathbf{n-k}})$ for player i, and by $NE_k(y)$ the set of the Nash equilibrium profiles. By NE_k we mean the correspondence mapping to $y \in X$ the

[1]The Stackelberg assumption could imply that either non-signatories or signatories behave as Stackelberg leaders. However, in the proposed model non-signatories are assumed to act as singletons and only Stackelberg leadership of the signatories (which basically act as a single player) has been assumed in the literature so far. In IEA context it may be argued that signatories are better informed than non-signatories about emission levels in other countries since they coordinate their environmental policies within an IEA (Finus 2001, Chapter 13).

set $NE_k(y) \in X^k$. If the game $\Gamma_k(y)$ has a unique Nash equilibrium for any y, say $(\eta_1(y),...,\eta_k(y))$, the signatories $P_{k+1},...,P_n$ maximize the joint profit function and solve the problem

$$\max_{y\in X} \sum_{j=k+1}^{n} f_j(\eta_1(y),...,\eta_k(y),y_{\mathbf{n-k}}). \tag{1}$$

Definition 1. *A vector* $x(k) = (\eta_1(\xi),...,\eta_k(\xi),\xi_{\mathbf{n-k}}) \in X^n$ *such that* ξ *solves the problem* (1) *is called a partial cooperative equilibrium of the game* Γ *where* $n-k$ *players sign the agreement.*

The definition of partial cooperative equilibrium has been given by Mallozzi and Tijs (2006) for symmetric potential games by considering the unique symmetric Nash equilibrium for non-signatories, together with an existence result. In this case the symmetry of the Nash equilibria allows to avoid coordination problems (Cooper and John 1988).

The uniqueness assumption of the Nash equilibrium $(\eta_1(y),...,\eta_k(y))$ of the game $\Gamma_k(y)$ not always occurs even in the symmetric case, as it is shown in the following example.

Example 1. Let us consider $n = 4$, $X_i = [0,1], i = 1,2,3,4$ and the following payoffs

$$f_i(x) = (x_1+x_2+x_3+x_4)^2 - 2x_i, \quad i = 1,2$$

$$f_3(x) = (x_1+x_2+x_3+x_4)^2 - 16x_3^2$$

$$f_4(x) = (x_1+x_2+x_3+x_4)^2 - 24x_4^2.$$

If two of the four players cooperate P_3 and P_4 by choosing $x_3 = x_4 = y$, the rest of the players choose a Nash equilibrium of the two-player game with payoffs

$$f_i(x_1,x_2,y,y) = (x_1+x_2+2y)^2 - 2x_i,\ i = 1,2.$$

For $0 \le y \le 1/4$ there are two Nash equilibria profiles $(0,0),(1,1)$ and for $y > 1/4$ there is a unique Nash equilibrium profile $(1,1)$.

We deal now with the case where the non-signatories have multiple equilibria for a given decision of the signatories. Let us suppose that there is a rule to choose in the set of the non-signatories Nash equilibrium set, namely a selection of the correspondence NE_k, that is a function

$$\lambda : y \in X \mapsto (\lambda_1(y),...,\lambda_k(y)) \in NE_k(y).$$

This is a way to choose a profile in the set $NE_k(y)$ for any $y \in X$. Then, the signatories $P_{k+1},...,P_n$ solve the problem

$$\max_{y\in X} \sum_{j=k+1}^{n} f_j(\lambda_1(y),...,\lambda_k(y),y_{\mathbf{n-k}}). \tag{2}$$

Definition 2. *A vector* $x(k)=(\lambda_1(\xi),...,\lambda_k(\xi),\xi_{\mathbf{n-k}})\in X^n$ *such that* ξ *solves the problem* (2) *is called a* λ*-partial cooperative equilibrium of the game* Γ *where* $n-k$ *players sign the agreement.*

Under the usual assumptions, common in practical situations, X compact subsets of Euclidean spaces and f_i continuous functions on X^n for all $i=1,..,n$, if there exists an upper semi-continuous selection λ of the correspondence NE_k, there is a λ-partial cooperative equilibrium with level of *non*-cooperation k.

In the following the special case of the max-selection is investigated.

2.1 Max-selection

Let us consider a partially ordered set X that is a set X on which there is a binary relation $\preceq$ that is reflexive, antisymmetric and transitive. Recall that the function $f_i(y_i,x_{-i})$ has increasing differences in (y_i,x_{-i}) on $X\times X^{n-1}$ for all i if for all $y_i\in X$ and $x'_{-i},x''_{-i}\in X^{n-1}$ with $x'_{-i}\prec x''_{-i}$, the function $f_i(y_i,x''_{-i})-f_i(y_i,x'_{-i})$ is increasing in y_i (Vives 1999). If f is a differentiable function on $\mathscr{R}^n$, then f has increasing differences on $\mathscr{R}^n$ if and only if $\partial f/\partial x_i$ is increasing in x_j for all distinct i and j and all x. If f is a twice differentiable function on $\mathscr{R}^n$, then f has increasing differences on $\mathscr{R}^n$ if and only if $\partial^2 f/\partial x_i\partial x_j\geq 0$, for all distinct i and j.

Fix now a level of *non*-cooperation k in the game $\Gamma=\langle n;X;f_1,...,f_n\rangle$.

Proposition 1. *Let* X *be a closed real interval and* $f_1,...,f_n$ *continuous functions on* X^n. *If* $f_i(y_i,x_{-i})$ *has increasing differences in* (y_i,x_{-i}) *on* $X\times X^{n-1}$ *for all* i*, then the set* $NE_k(y_{n-k})\neq\emptyset$ *for any* y *and a greatest and a least equilibrium point exist.*

Proof. The game $\Gamma_k(y)$ turns out to be a supermodular game (Vives 1999) and by Topkis' theorem the set of the Nash equilibrium profiles is a nonempty complete lattice with respect to the natural partial ordering $\preceq$ in $\mathscr{R}^k$. □

By using Proposition 1 we consider the following *max-selection*

$$\overline{\lambda}(y)=\max\{\eta=(\eta_1,...,\eta_k):\eta\in NE_k(y)\}.$$

Proposition 2. *Let* X *be a closed real interval and* $f_1,...,f_n$ *continuous functions on* X^n. *If* $f_i(y_i,x_{-i})$ *has increasing differences in* (y_i,x_{-i}) *on* $X\times X^{n-1}$ *for all* i*, for each* $k=2,...,n$*, the function* $\overline{\lambda}$ *is upper semi-continuous on* X.

Proof. Since $f_1,...,f_n$ are continuous functions on X closed real interval, the Nash equilibrium correspondence is sequentially closed at $y\in X$ (Aubin and Frankowska, 1990), i.e. for any sequence (y_m) of X converging to $y\in X$ and any sequence (η_m) of X^k converging to $\eta\in X^k$ such that $\eta_m\in NE_k(y_m)$ for all $m\in N$, we have $\eta\in NE_k(y)$.

Let us consider $y\in X$ and a sequence (y_m) converging to y. For any $m\in\mathscr{N}$, the sequence $\overline{\lambda}(y_m)\in NE_k(y_m)$ is bounded. So, there exists a subsequence of $\overline{\lambda}(y_m)$ converging to $l=\limsup_m\overline{\lambda}(y_m)$. Since NE_k is closed, $l\in NE_k(y)$ and by definition of $\overline{\lambda}$ we have $l\leq\overline{\lambda}(y)$, i.e. $\limsup_m\overline{\lambda}(y_m)\leq\overline{\lambda}(y)$, i.e. λ is upper semi-continuous at y. □

Let us note that in Example 1, for $k=2$, the max-selection is the constant function mapping to any $y \in X$ the Nash equilibrium pair $(1,1)$.

As a consequence of Proposition 1 and Proposition 2 we have the following existence theorem.

Theorem 1. *Let X be a closed real interval and $f_1,...,f_n$ continuous functions on X^n. If $f_i(y_i,x_{-i})$ has increasing differences in (y_i,x_{-i}) on $X \times X^{n-1}$ for all i, for each $k=2,...,n-1$, there exists at least a $\overline{\lambda}$-partial cooperative equilibrium $x(k)=(\overline{\lambda}_1(\xi),...,\overline{\lambda}_k(\xi),\xi_{\mathbf{n-k}}) \in X^n$ of the game Γ.*

Remark 1. It is easy to find not upper semicontinuous selections. In Example 1, the *min-selection*

$$\underline{\lambda}(y)=\min\{\eta=(\eta_1,...,\eta_k) : \eta \in NE_k(y)\}$$

is a selection of the Nash equilibrium correspondence that is not u.s.c.: for $0 \le y \le 1/4$ we have $\underline{\lambda}(y)=(0,0)$ and for $1/4<y\le 1$ we have $\underline{\lambda}(y)=(1,1)$. The two cooperating players P_3 and P_4 have to jointly maximize the function $(f_3+f_4)(1,1,y,y)=2(2+2y)^2-16y^2-24y^2$ for $y>1/4$, $(f_3+f_4)(0,0,y,y)=2(2y)^2-16y^2-24y^2$ for $0\le y\le 1/4$, that is not u.s.c. at $y=1/4$ and a $\underline{\lambda}$-partial cooperative equilibrium does not exist.

3. Public goods game

Let us consider a situation where n agents interact to consume a good having a "public" character, for example roadways (Mas-Colell et al. 1995, Ray and Vohra 1997, Yi 1997, Batina and Ihori 2005). There are n identical consumers of one public good. Each consumer can provide $x_i \in X$ (X real closed interval) units of the public good at cost $C_i(x_i)$ and enjoys a benefit depending on the total amount G of the public good $B_i(G)$. The usual assumptions in the twice differentiable case are the following for each i:

$$C_i' \ge 0, B_i' \ge 0, C_i'' \ge 0, B_i'' \le 0$$

The consumer i's utility is $f_i(x_i,G)=B_i(G)-C_i(x_i)$. The quantity G represents the public consumption. The simplest version of the public goods game assumes a linear technology constraint, which transforms one unit of private good into one unit of public good, $G=\sum_{i=1}^n x_i$.

The strategic form game $\Gamma^{PG}=\langle n;X;B_1,..,B_n,C_1,..,C_n\rangle$ is called public goods game. Public goods games are examples of aggregative games (Corchon, 1994), i.e. the payoffs depend only on individual strategies and an aggregate of all strategies $\sum_{i=1}^n x_i$. In this case $g(x_1,...,x_n)=\sum_{i=1}^n x_i$ is called aggregator function; usually g is a continuous increasing function of $(x_1,...,x_n)$.

In order to apply Theorem 1, if we require for the game Γ^{PG} the increasing differences property, we would have $\partial f_i^2/\partial x_i \partial x_j = B_i'' \ge 0$ against the usual assumption $B_i'' \le 0$. So that in the game Γ^{PG} we would have linear benefit function B_i for each i and the utility functions separable in all variables.

A way out to have increasing differences property in a public goods game is to deal with non-linear technology constraint. We assume $G=\left(\sum_{i=1}^n x_i\right)^2$. In this case we

can combine multiplicity and supermodularity in a special class of public good games, namely public goods games with linear benefit function B_i for each i.

Let Γ^{PG} be a public goods game with $G = \left(\sum_{i=1}^{n} x_i\right)^2$, B_i linear for each i and $C_i'' \geq 0$ for each i. The game satisfies all assumptions of the Theorem 1 and we have the following existence result.

Proposition 3. *For any level of non-cooperation k there exists at least a $\overline{\lambda}$-partial cooperative equilibrium $x(k) = (\overline{\lambda}_1(\xi), ..., \overline{\lambda}_k(\xi), \xi_{\mathbf{n}-\mathbf{k}}) \in X^n$ of the game Γ^{PG}.*

The game in Example 1 is a public goods game with $G = \left(\sum_{i=1}^{n} x_i\right)^2$, $B_i(t) = t, i = 1, ..4$, $C_1(t) = C_2(t) = 2t$, $C_3(t) = 16t^2$, $C_4(t) = 24t^2$. The two cooperating players P_3 and P_4 jointly maximize the function

$$(f_3 + f_4)(\overline{\lambda}_1(y), \overline{\lambda}_2(y), y, y) = 2(2+2y)^2 - 16y^2 - 24y^2$$

being $(\overline{\lambda}_1(y), \overline{\lambda}_2(y)) = (1,1)$ for any y. The $\overline{\lambda}$-partial cooperative equilibrium is then $(1,1,1/4,1/4)$.

For any $y \in X$ and any $i > k$, since $B_i' \geq 0$ we have

$$\max_{\eta \in NE_k(y)} B_i\Big(\sum_{j=1}^{k} \eta_j + (n-k)y\Big)^2 - C_i(y) = B_i\Big(\sum_{j=1}^{k} \overline{\lambda}_j(y) + (n-k)y\Big)^2 - C_i(y),$$

where $\overline{\lambda}$ is the max-selection of the Nash equilibrium correspondence. So that the max-selection $\overline{\lambda}$ is the best choice from the signatory point of view.

4. Conclusion

We investigated a partial cooperation approach between a group of the players who cooperate by choosing the same strategy; the rest of the players play a Nash equilibrium game. In the case where the set of the possible Nash equilibria for the non-cooperating players is not a singleton, we proposed a definition of partial cooperative equilibrium depending on a selection choice in the set of equilibria. We presented some results in the max-selection case. It could be interesting to develop further the study of λ-partial cooperation by using different selection choices besides the max-selection, in order to have existence of the λ-partial cooperative equilibrium and also properties useful in concrete applications as the public goods game.

In the proposed model the cooperative aspect of the partial cooperative equilibrium concept has been formalized by mean of the joint (common) strategy of the signatories. This model fits in the IEA framework where the signatory players are those countries signing the agreement and choosing the joint emission level (Finus 2001). It is possible to give a more general definition of partial cooperative equilibrium by assuming that the signatories choice is a vector $y_{\mathbf{n}-\mathbf{k}} = (y_{k+1}, ..., y_n)$ having different components and maximize again their aggregate welfare

$$\max_{y_{\mathbf{n}-\mathbf{k}} \in X^{n-k}} \sum_{j=k+1}^{n} f_j(\eta_1(y_{\mathbf{n}-\mathbf{k}}), ..., \eta_k(y_{\mathbf{n}-\mathbf{k}}), y_{\mathbf{n}-\mathbf{k}}),$$

where the vector $(\eta_1(y_{\mathbf{n-k}}), ..., \eta_k(y_{\mathbf{n-k}}))$ is the Nash equilibrium chosen by the non-signatories. The definition of λ-partial cooperative equilibrium and existence results could be given in this case similarly to those proved in the paper.

A deeper formalization of the behavior of the cooperating group could be given in terms of other possible solution concepts used in a Cooperative Game Theory setting, for example the Shapley value. Further problems arise in these more general cases, for example the redistribution of utility between cooperating players and the stability of the coalition. This will be developed in a future paper.

Acknowledgment The authors thank the referees for their interesting comments.

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