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Can CDFC and MLRP Conditions Be Both Satisfied for a Given Distribution?

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Abstract In principal-agent problem, the first-order approach is frequently used. To insure the validity of the approach the Monotone Likelihood Ratio Property and the Convexity of the Distribution Function Condition are requested. While the former property is satisfied by most of the distributions, this is not the case for the second property. We present two families of distributions for which the properties are satisfied. The first family includes as special cases the distributions that were previously introduced by various authors. The second family includes new distributions.

Keywords Monotone Likelihood Ratio Property, Convex Distribution Function Condition, incentive contract, first-order approach, moral hazard
JEL classification D3, D8

1. Introduction

The principal-agent problem is usually formulated in the following form: the principal maximizes his utility under two constraints—a participation constraint requires that the agent accepts to participate and an incentive constraint explains that the agent chooses an effort that maximizes his utility. Under the two conditions, in the continuous formulation of this problem, the second constraint may be replaced by a first-order condition. This modified formulation, usually called “the first-order approach”, has been justified by various authors (Mirrlees 1999; Rogerson 1985; Jewitt 1988; Carlier and Dana 2005).

Many authors have highlighted the importance of two kinds of assumptions about the output probability distributions with respect to effort that insure the validity of the first-order approach. The first one is the Monotone Likelihood Ratio Property (MLRP), which provides payment increasing with the result. The second one is the Convexity of the Distribution Function Condition (CDFC) which imposes the convexity with respect to effort of the probability distribution of the result. While the former property (MLRP) is satisfied by most of the distributions, this is not the case for the second property (CDFC).

In the literature a few articles propose distributions which satisfy the CDFC (Grossman and Hart 1983). Often, these distributions have bounded support (Rogerson 1985; LiCalzi and Spaeter 2003). Holmstrom (1984) proposed a class of distributions with

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unbounded support. Benassi (2011) proposed convexifying transformations in order to produce distributions satisfying the CDFC property. The interest of finding new families of distribution lies in the fact that for two classes of relatively similar distributions, one may satisfy the property and and the other not. A researcher may need to specify a unimodal or multimodal distribution, depending on the empirical data. With a wider choice of distribution classes, it is easier to fit a suitable distribution with a given empirical distribution.

After a description of the principal-agent formulations using the first-order approach, we present two major families of distributions (with bounded and unbounded support) candidate for the verification of MLRP and CDFC properties. We indicate the conditions necessary to satisfy these conditions for the different classes of distributions. Classes of the first family generalise the classes given in Holmstrom (1984), Rogerson (1985), LiCalzi and Spaeter (2003). We introduce a second family of a new type and we give examples of distributions that satisfy or do not satisfy the properties.

2. The first-order approach in principal-agent problems

We consider the classical continuous formulation of the principal-agent problem: the principal values the effort a. This effort leads to a result x which is observable by the principal: x is a random variable following the cumulative distribution $H(x;a)$, with density function $h(x;a)$, conditioned by the effort a. The working effort a chosen by the agent is not observable by the principal. The effort and the result are supposed to vary continuously.

We consider the context of a classic risk averse agent with utility function $U(\cdot)$ and a principal risk averse with utility function $V(\cdot)$. The principal and the agent maximize their own expected utility. The transfer monetary function $t(\cdot)$ and the optimal effort a^* are obtained by solving the following program:

$$\max_{t(\cdot),a} \int_x V(x-(1+\gamma)t(x))dH(x;a)$$

$$\begin{aligned} \text{s. t.} \quad & \text{participation constraint:} && \int_x (U(t(x))-a^*)dH(x;a^*) \geq U_0, \\ & \text{incentive constraint:} && \max_a \int_x (U(t(x))-a)dH(x;a). \end{aligned}$$

With the first order approach the incentive constraint is replaced by its first order condition:

$$\int_x U(t(x))dH_a(x;a^*) = 1.$$

The link between the initial problem and the modified problem has been studied by many authors (Mirrlees 1999; Rogerson 1985; Jewitt 1988; Carlier and Dana 2005). These authors have shown that if the MLRP and CDFC properties hold:

MLRP: the likelihood ratio $\dfrac{h(x;a)}{h(y;a)}$ is increasing in a for all $x>y$, i.e. $\dfrac{h_a}{h}(x;a)$ is increasing in x (i.e. $\dfrac{\partial\left(\frac{h_a}{h}(x;a)\right)}{\partial x} \geq 0$ for all x if H is twice differentiable).

CDFC: the distribution function is convex in a for all x, i.e. H_a is increasing in a (i.e. $H_{aa}(x) > 0$ for all x if H is twice differentiable).

Then it is valid to replace the initial problem by the modified problem. These conditions satisfied, the authors infer the existence of monotonous solutions.

While the MLRP property is satisfied by most of the distributions, it is not the case for the CDFC property. In the literature, we find, with specific conditions:

- the convex combination of two distributions on the outcome x in Holmstrom (1984): $H(x;a) = \theta(a)F(x) + (1-\theta(a))G(x)$;
- the distribution function $H(x;a) = x^a$ (Rogerson 1985);
- the two classes in LiCalzi and Spaeter (2003): $H(x;a) = x + \beta(x)\gamma(a)$ and $H(x;a) = \delta(x)e^{\beta(x);\gamma(a)}$
- Benassi (2011) proposed convexying mappings which transform any given distribution into one satisfying the (CDFC) condition.

At first sight, some distributions of the first class in LiCalzi and Spaeter (2003) are a convex combination of two distributions so they are included in the class in Holmstrom (1984). Other distributions are particular cases which can be included in more general classes. In order to clarify the situation, we propose a classification in two families: the first family is itself divided into three classes. Concerning the two families: the "outer" and "inner" characters of the distributions are related to the dependency of the distribution with respect to the effort a.

3. The first family of "outer" distributions

In this first family, the cumulative distribution is separable in the result x and the effort a and is generated by distributions of the result x.

Consider $F(\cdot)$, respectively $G(\cdot)$, a cumulative distribution with density function f, respectively g, of the result x, on the support X and $\theta(\cdot)$ a function with support A and value in $[0,1]$. Due to the separation of the result x and the effort a, no hypothesis on the support X (bounded or unbounded) is necessary to insure the support X is independent of the effort a. We distinguish three classes.

3.1 A class based on a distribution

The following class generalises the distribution function in Rogerson (1985).

Proposition 1. *Let F be a distribution function with support X and $\theta(\cdot)$ with support A and value in $[\underline{\theta}, \overline{\theta}]$, $\underline{\theta} > 0$. The distribution function*

$$H(x;a) = F(x)^{\theta(a)}$$

satisfies the MLRP and CDFC properties if the following conditions hold:

(i) *F is continuously differentiable;*

(ii) *θ is a continuously differentiable increasing and concave function in a.*

Proof. By construction, H is a cumulative distribution on X. H satisfies the MLRP property because the condition

$$\frac{\partial\left(\frac{h_a}{h}(x;a)\right)}{\partial x} = \theta'(a)\frac{f(x)}{F(x)} \geq 0$$

is satisfied. Moreover $H_a(x;a) = \theta'(a)\log(F(x))F(x)^{\theta(a)}$. From $\log(F(x)) < 0$ and $\theta'(a)F(x)^{\theta(a)}$ decreasing in a, we deduce that H_a is increasing in a, hence H satisfies the CDFC property. □

No condition is imposed to the support X of F. X may be bounded or unbounded. The distribution function $H(x;a) = x^a$ in Rogerson (1985) belongs to this class with $F(x) = x$ and $\theta(a) = a$ on bounded support.

Alternatively, properties are inherited from distribution F to distribution H. This class of distribution is included in a larger class proposed by Benassi (2011): by appropriate transformation ϕ, any given distribution F of the result may generate a conditional distribution H which satisfies the CDFC and the MLRP conditions, $H(x,a) = \phi(F(x),a)$. The conditions on ϕ to be checked are given by the author as well as an example of transformation: $\phi(y,a) = ((ae)^y - 1)/(ae-1)$. Let us point out that, for our class $\phi(y,a) = y^{\theta(a)}$.

Corollary 1. *Let F be a distribution satisfying the hypotheses of Proposition 1:*

(i) *if F is log-concave then H is log-concave;*

(ii) *if $\underline{\theta} \geq 1$ and f is log-concave then h is log-concave;*

(iii) *the stochastic dominance is preserved in the following sense: if distribution F_1 stochastically dominates F_2, then the corresponding distribution H_1 stochastically dominates H_2.*

Proof.

(i) The result is deduced from: $\log H = \theta \log F$ and $\theta > 0$.

(ii) From Bagnoli and Bergstrom (2005), f log-concave implies that F is log-concave and the result is deduced from: $\log h = \log f + (\theta - 1)\log F + \log\theta$ and $\theta > 1$.

(iii) The result is deduced form $(\log F_1)' \geq (\log F_2)'$ and $(\log H_i)' = \theta(\log F_i)'$, $i = 1,2$.

□

3.2 Class of the convex combination of two distributions

We recall the convex combination proposed by Holmstrom (1984) (see also Rogerson 1985; Hart and Holmstrom 1987; Sinclair-Desgagné 1994):

Proposition 2. (Holmstrom 1984, Hart and Holmstrom 1987) *Let $X \subseteq R^+$, $A = R^+$, F and G be two distribution functions with support X, $\theta(\cdot)$ with support A and value in* $[0,1]$. *The distribution function*

$$H(x;a) = \theta(a)F(x) + (1-\theta(a))G(x)$$

satisfies the MLRP and CDFC properties if the following conditions hold:

(i) *$f(x)$ and $g(x)$ are strictly positive for all x in X, F and G are such that: $F(x) \leq G(x)$ and f/g is nondecreasing on X.*

(ii) *$\theta(\cdot)$ is a twice continuously differentiable increasing and concave function such that $\theta(0) = 0$, $\lim_{a\to+\infty} \theta(a) = 1$.*

We remark that Proposition 2 is also true for bounded A and θ continuously differentiable. With the notation of Proposition 1 of LiCalzi and Spaeter (2003), if we denote: $F(x) = x - \beta(x), G(x) = x$ and $\theta(a) = -\gamma(a)$, from condition (i) of Proposition 1, $F(x)$ is actually a cumulative distribution. From Proposition 2, distributions of the first class of LiCalzi and Spaeter are a particular case of convex combination of two distributions.

Similar log-concavity and stochastic dominance properties to those obtained in Corollary 1 can be deduced.

3.3 Class of the product of two distributions

The following class of the distributions defined as the product of two distributions on the result generalises the second class in LiCalzi and Spaeter (2003).

Proposition 3. *Let F and G be two distribution functions with the same support X, $\theta(\cdot)$ with support A and value in* $[0,1]$. *The distribution function*

$$H(x;a) = F(x)^{\theta(a)}G(x)^{1-\theta(a)}$$

(or the equivalent form $H(x;a) = \widetilde{F}(x)^{\theta(a)}G(x)$ with $\widetilde{F}(x) = \dfrac{F(x)}{G(x)}$) satisfies the MLRP and CDFC properties if the following conditions hold:

(i) *F and G are continuously differentiable, $f(x)$ and $g(x)$ are strictly positive for all x in X.*

(ii) *F stochastically dominates G (i.e. $\dfrac{F(x)}{F(y)} \leq \dfrac{G(x)}{G(y)}$ for all $x \leq y$, i.e. $\dfrac{f(x)}{F(x)} \geq \dfrac{g(x)}{G(x)}$).*

(iii) *There exists a non-negative and decreasing function* $k(x) : X \to [0,1]$ *such that* $\dfrac{g(x)}{G(x)} = k(x)\dfrac{f(x)}{F(x)}$.

(iv) θ *is a continuously differentiable increasing and concave function in a.*

Proof. By construction, H is a distribution function on X. $\dfrac{h_a}{h}(x;a)$ may be written:

$$\frac{h_a}{h}(x;a) = \theta'(a)[\log\widetilde{F}(x) + \frac{1}{\theta(a) + \frac{(\log G)'}{(\log\widetilde{F})'}(x)}]$$

From (ii) $\widetilde{F}(x)$ is increasing and from (iii) $\dfrac{(\log G)'}{(\log\widetilde{F})'} = 1/(1 - \dfrac{(\log G)'}{(\log F)'}) = 1/(1-k(x))$ is decreasing on X. Combined with (iv) we deduce that $\dfrac{h_a}{h}(x;a)$ is increasing in x, hence H satisfies the MLRP property.

Moreover $H_a(x;a) = G(x)\theta'(a)\log(\widetilde{F}(x))\widetilde{F}(x)^{\theta(a)}$. Due to $\log\widetilde{F}(x) < 0$ and (iv), H_a is increasing in a, then H satisfies the CDFC property. □

The link with the second class in LiCalzi and Spaeter (2003) is obtained if we note $F(\cdot) = \delta(\cdot)e^{\beta(\cdot)}$ and $G(\cdot) = \delta(\cdot)$. From Proposition 3 we deduce that we can extend this class to distributions with non-concave $\delta(\cdot)$ or non-convex $\beta(\cdot)$.

The conditions in Proposition 3 are more restrictive than conditions in Proposition 2 for the distributions F and G; conditions (ii) and (iii) of Proposition 3 imply condition (i) of Proposition 2. Contrary to the previous classes, log-concavity properties of distribution H may not be inherited form distributions F and G.

4. The second family of "inner" distributions

In this second family, distributions are generated by a distribution on the ratio of the result x and a function of the effort a, $x/\theta(a)$.

Let $H(x;a) = F(\frac{x}{\theta(a)})$ where F is a distribution function with unbounded support. An unbounded support insures a support X independent of the effort a.

Proposition 4. *Let* $X = R^+$, F *be a distribution function with support* X *and* $\theta(\cdot)$ *with support* A *and value in* $[\underline{\theta}, \overline{\theta}]$, $\underline{\theta} > 0$. *The distribution function*

$$H(x;a) = F\left(\frac{x}{\theta(a)}\right)$$

satisfies the MLRP and CDFC properties if the following conditions hold:

(i) F *is twice continuously differentiable,* $f(z)$ *is strictly positive for all* $z > 0$.

(ii) $s(z) = z\dfrac{f'}{f}(z)$ *is decreasing in z.*

(iii) *there exists* $K < 2$ *such that* $z^{2-K} f(z)$ *is increasing in* z.

(iv) θ *is a continuously differentiable increasing function in the effort a.*

(v) *For the same K,* $\dfrac{\theta'(a)}{\theta(a)^K}$ *is decreasing in a and it exists* $k < K$ *such that* $\dfrac{\theta'(a)}{\theta(a)^k}$ *is increasing in a.*

Conversely, if the condition (ii) does not hold, then distribution H does not satisfy the MLRP condition. If the condition (iii) is replaced by condition (vi)*:*

(vi) $z^{2-k} f(z)$ *is decreasing in z on a proper interval,*

then distribution H does not satisfy the CDFC condition.

Proof. By construction, H is a distribution function on $X = R^+$. Let $z_a = \dfrac{x}{\theta(a)}$, then

$$\frac{h_a}{h}(x;a) = -\frac{\theta'(a)}{\theta(a)}[1 + z_a \frac{f'}{f}(z_a)].$$

As θ is increasing and $z\dfrac{f'}{f}(z)$ is decreasing, we deduce the MLRP property.

Moreover $H_a(x;a) = -\dfrac{\theta'(a)}{\theta(a)} z_a f(z_a) = -\dfrac{\theta'(a)}{\theta(a)^K} z_a^{2-K} f(z_a) x^K$. From the positivity and the decreasing of $\theta'(a)/\theta(a)^K$ and $z_a^{2-K} f(z_a)$ in a, the CDFC property is satisfied.

Conversely, if (ii) does not hold, the result is trivial. If (iii)' holds, $z_a^{2-k} f(z_a)$ is positive and increasing in a on a proper interval and from (v) $\theta'(a)/\theta(a)^k$ is positive and increasing in a, then from $H_a(x;a) = -\dfrac{\theta'(a)}{\theta(a)^k} z_a^{2-k} f(z_a) x^k$, H_a is decreasing in a on a proper interval. □

The conditions with k and K in (v) are equivalent to the following:

- $\log \theta(a)$ $(K = 1)$ or $(1-K)\theta(a)^{1-K}$ $K < 1$ is concave if $K < 1$ and $(1-k)\theta(a)^{1-k}$ is convex if $k < 1$.
- $k\theta'(a)^2 \leq \theta''(a)\theta(a) \leq K\theta'(a)^2$ for all a if θ is twice continuously differentiable.

Example 1. *The following distributions F allow to generate distributions H which belong to the second family:*

- *the Frechet distribution* $F(z) = e^{-\frac{q}{z^p}}$,
- *the Log-logistic distribution* $F(z) = 1 - \dfrac{1}{1+qz^p}$.

The obtained conditions in Proposition 4 are complex but it is possible to deduce a sufficient condition on distributions F that permit to predict that no distributions H of the second family can ge generated. The possibility to generate is linked with the behavior of the function $s(\cdot)$. From (ii), $s(z)$ must be decreasing but due to (iii), $s(z)$ must also not too much decreasing, thus we obtain the following characterization:

Corollary 2. *If θ verifies assumptions* (iv) *and* (v) *in Proposition 4, $f(z)$ is strictly positive for all $z > 0$ and $\liminf_{z->+\infty} s(z) = -\infty$ with $s(z) = z\frac{f'}{f}(z)$, then distribution F (associated with θ) does not generate a distribution H which belongs to the second family.*

Proof. For a large enough z value: $[z^{2-k}f(z)]' = (2-k+z\frac{f'}{f}(z))z^{1-k}f(z) < 0$ for all $k < 2$. □

Example 2. *From Corollary 2, with assumptions (iv) and (v) for θ, the following distributions F (associated with θ) do not generate distributions H that belong to the second family:*

- *the Weibull distribution $F(z) = 1 - e^{-qz^p}$ with $s(z) = -qpz^p$,*
- *the Lognormal distribution $f(z) = \frac{1}{\sqrt{2\pi}\sigma z}e^{-\frac{1}{2}(\frac{\log z}{\sigma})^2}$ with $s(z) = -1 - \frac{\log z}{\sigma^2}$,*
- *the Gamma distribution $f(z) = \frac{q^{p+1}z^p}{\Gamma(p+1)}e^{-qz}$ with $s(z) = p - qz$.*

From the Corollary 2 and the given examples, we deduce that, for a distribution to generate a function of this family, it is necessary that the density does not tend toward zero too fast.

Log concavity and stochastic dominance properties are inherited from distribution F to distribution H.

Corollary 3. *Let F a distribution satisfying the hypotheses of Proposition 4:*

(i) *if F is log-concave then H is log-concave;*

(ii) *if f is log-concave then h is log-concave;*

(iii) *the stochastic dominance is preserved in the following sense: if distribution F_1 stochastically dominates F_2, then the corresponding distribution H_1 stochastically dominates H_2.*

Proof.

(i) The result is deduced from: $(\log H)''_{x^2} = \frac{1}{\theta^2}(\log F)''_{z^2}$.

(ii) The result is deduced from: $(\log h)'_x = \frac{1}{\theta}(\log f)'_z$ with $\theta > 0$.

(iii) The result is deduced form $(\log F_1)'_z \geq (\log F_2)'_z$ and $(\log H_i)'_x = \frac{1}{\theta}(\log F_i)'_z, i = 1,2$ with $\theta > 0$. □

Remark 1. *For distributions H of this family, the mean value and the variance are link to those of the distribution F: $E(X) = \theta(a)E(Z)$ and $\sigma(X) = \theta(a)\sigma(Z)$ with the random variable X and Z of distributions F and H. More generally $E(X^p) = \theta(a)^p E(Z^p)$. Hence H and F have the same coefficient of variation, independently of the value of the effort a.*

5. Discussion

The two families have a non-null intersection. The distribution $H(x;a) = e^{-q\frac{\alpha(a)}{x^p}}$, with $p > 0$ and α concave, belongs to the two families and was generated by the distribution $F(z) = e^{-\frac{q}{z^p}}$: with $\theta(a) = \alpha(a)$ in the first class of the first family, with $\theta(a) = \alpha(a)^{\frac{1}{p}}$ in the second family ($K = 1 - p$). This distribution is the unique distribution in the intersection. The proof of this uniqueness is given in Appendix.

For the distributions of Proposition 2 and 3, even if F and G are unimodal, H can be bimodal.

For the distributions of Proposition 4 the properties of unimodality are inherited from distribution F to distribution H. Except for the class of product of distribution, log concavity and stochastic dominance properties are inherited from distributions F and G to distribution H.

In the two families, the distributions are generated by distributions of one variable: the result in the first family, a function of the result and the effort in the second family.

6. Conclusion

We exhibit two families of distributions which satisfy the MLRP and CDFC properties. The first family (in which the result and the effort are separable) contains and generalises all the previously known distributions. The family is divided into several classes. A second family contains new distributions and a characterisation of these distributions is given. We show that the considered conditions are compatible for numerous distributions.

In addition, we give some examples of unimodal distributions that can help researchers to choose a distribution for their works.

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References

Bagnoli, M. and Bergstrom, T. (2005). Log-Concave Probability and Its Applications. *Economic Theory*, 26(2), 445–469.

Benassi, C. (2011). A Note on Convex Transformations and the First Order Approach. Rimini, the Rimini Centre for Economic Analysis, Working paper No. 6.

Carlier, G. and Dana, R. A. (2005). Existence and Monotonicity of Solutions to Moral Hazard Problems. *Journal of Mathematical Economics*, 41, 826–843.

Grossman, S. J. and Hart, O. D. (1983). An Analysis of the Principal-Agent Problem. *Econometrica*, 51, 7–45.

Hart, O. and Holmstrom, B. (1987). The Theory of Contracts. In Bewley, T. (ed.), *Advances in Economic Theory: Fifth World Congress*. Cambridge, Cambridge University Press, 71–156.

Holmstrom, B. (1984). A Note on the First-Order Approach in Principal-Agent Models. Mimeo.

Jewitt, I. (1988). Justifying the First-Order Approach to Principal-Agent Problems. *Econometrica*, 56(5), 1177–1190.

LiCalzi, M. and Spaeter, S. (2003). Distributions for the First-Order Approach to Principal-Agent Problems. *Economic Theory*, 21, 167–173.

Mirrlees, J. A. (1999). The Theory of Moral Hazard and Unobservable Behavior: Part I. *Review of Economic Studies*, 66, 3–21.

Rogerson, W. P. (1985). The First-Order Approach to Principal-Agent Problems. *Econometrica*, 53, 1357–1367.

Sinclair-Desgagné, B. (1994). The First-Order Approach to Multi-Signal Principal-Agent Problems. *Econometrica*, 62(2), 459–465.

Appendix: Intersection of the two families

Assume that $H(x;\theta)$ belongs to the two families, hence it exists $F_1, F_2, \theta_1, \theta_2$ such that

$$H(x;a) = F_1(x)^{\theta_1(a)} = F_2\left(\frac{x}{\theta_2(a)}\right).$$

Then $\log F_1(x) = \dfrac{1}{\theta_1(a)} \log F_2\left(\dfrac{x}{\theta_2(a)}\right)$. The right side of this equation is independent of a then the derivative of the right side with respect to a is equal to zero:

$$-\frac{\theta_1'(a)}{\theta_1^2(a)} \log F_2\left(\frac{x}{\theta_2(a)}\right) - \frac{\theta_2'(a)}{\theta_1(a)\theta_2^2(a)} x \frac{f_2}{F_2}\left(\frac{x}{\theta_2(a)}\right) = 0$$

We deduce, using the variables $z = x/\theta_2(a)$ and a:

$$\frac{F_2(z)\log F_2(z)}{zf_2(z)} = -\frac{\frac{\theta_2'}{\theta_2}(a)}{\frac{\theta_1'}{\theta_1}(a)}$$

The left side is independent of a, the right side is independent of z, so the two sides are constant. As θ_1 and θ_2 are increasing in a we deduce the sign of the constant, $k > 0$, and:

$$F_2(z)\log F_2(z) = -kzf_2(z) \tag{A1}$$

$$\frac{\theta_2'}{\theta_2}(a) = k\frac{\theta_1'}{\theta_1}(a) \tag{A2}$$

From (A1): $(\log(-\log F_2))'(z) = \dfrac{(\log F_2)'}{\log F_2}(z) = -\dfrac{1}{kz}$ then by integration we deduce that it exists c such that $\log(-\log F_2(z)) = c - \dfrac{1}{k}\log(z)$ and $F_2(z) = e^{-\frac{e^c}{z^{1/k}}}$.

From (A2): $(\log\theta_2)'(a) = k(\log\theta_1)'(a)$ then by integration $\log\theta_2(a) = k\log\theta_1$ and $\theta_2(a) = \theta_1(a)^k$.

Then the distribution $H(x;a) = e^{-q\alpha(a)/z^p}$ is generated by $F_1(y) = F_2(y) = e^{-q/y^p}$ with $q = e^c$ and $p = \frac{1}{k}$ and belongs to the two families with $\theta_1(a) = \alpha(a)$ in the first family and $\theta_2(a) = \alpha(a)^{\frac{1}{p}}$ in the second family.