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Approval Voting without Faithfulness

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Abstract In this short paper, we analyze the implications of dropping the axiom of faithfulness in the axiomatization of approval voting, due to P. C. Fishburn. We show that a ballot aggregation function satisfies the remaining axioms (neutrality, consistency and cancellation) if and only if it is either a function that chooses the whole set of alternatives, or an approval voting, or a function that chooses the least approved alternatives.

Keywords Approval voting, faithfulness, inverse approval voting
JEL classification D71, D72

1. Introduction

There are a number studies related to axiomatization of approval voting (AV) in the literature (for a survey, see Xu 2010). Early results on this problem are due to Fishburn (1978a, 1978b). Fishburn (1978a) shows that AV is the only ballot aggregation function (BAF) satisfying the axioms of neutrality, consistency, cancellation and faithfulness, while Fishburn (1978b) axiomatizes AV with the axioms of neutrality, consistency and disjoint equality. Recently, Alós-Ferrer (2006) shows that one can drop the axiom of neutrality in the axiomatization of Fishburn (1978a).

The primary objective of this short paper is to investigate the implications of dropping the axiom of faithfulness in the axiomatization of Fishburn (1978a), hence to analyze its cutting power. Our main finding is that a ballot aggregation function satisfies the remaining axioms, namely, neutrality, consistency and cancellation, if and only if it is either a trivial function that chooses the whole set of alternatives at all profiles, or AV, or its inverse, that is a function that chooses the least approved alternatives (see Theorem 1).

2. Characterization

Let $\mathbb{N}$ denote the set of nonnegative integers. Let X be a (finite) set of alternatives and let S be the set of all permutations of X. For $\sigma \in S$ and $Y \subseteq X$, $\sigma(Y) \subseteq X$ is the image of Y under σ. A *ballot* B is a nonempty subset of X and let $\mathcal{B} = 2^X \setminus \{\emptyset\}$ be the set of all admissible ballots. Voters can cast any ballot, approving as many candidates as they want. A *voter response profile* is a function $\pi : \mathcal{B} \rightarrow \mathbb{N}$ such that $\pi(B)$ is the number of voters who cast ballot B. Let Π be the set of all possible voter response profiles,

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including the empty profile π^0 with $\pi^0(B) = 0$ for all $B \in \mathcal{B}$. A *ballot aggregation function* (BAF) is a correspondence f which assigns to every possible voter response profile $\pi \in \Pi$, a nonempty set of selected alternatives, $\emptyset \subsetneq f(\pi) \subseteq X$. A BAF is an *approval voting* (f^A) if

$$f^A(\pi) = \arg\max_{x \in X} \sum_{\{B \in \mathcal{B}: x \in B\}} \pi(B),$$

it is an *inverse approval voting* (f^{-A}) if

$$f^{-A}(\pi) = \arg\min_{x \in X} \sum_{\{B \in \mathcal{B}: x \in B\}} \pi(B),$$

and it is a *trivial* function (f^*) if $f^*(\pi) = X, \forall \pi \in \Pi$.

Given $x \in X$ and $\pi \in \Pi$, the number of voters who approve of x in π is given by

$$v(x, \pi) = \sum_{\{B \in \mathcal{B}: x \in B\}} \pi(B).$$

For any $B \in \mathcal{B}$, let $\pi_B \in \Pi$ denote the voter response profile with $\pi_B(B) = 1$ and $\pi_B(B') = 0$ for all $B' \neq B$, i.e. π_B consists of only one ballot B. When $B \in \mathcal{B}$ consists of a single element $x \in X$, we write π_x instead of $\pi_{\{x\}}$. For any $\pi, \pi' \in \Pi$, $\pi + \pi' \in \Pi$ is a voter response profile with $(\pi + \pi')(B) = \pi(B) + \pi'(B)$, $\forall B \in \mathcal{B}$, and whenever $\pi, \pi' \in \Pi$ are such that $\pi(B) = \pi'(B), \forall B \in \mathcal{B}$, we write $\pi = \pi'$.

A BAF satisfies

Neutrality: if $f(\pi \circ \sigma) = \sigma(f(\pi))$ for every $\sigma \in S$ and for every $\pi \in \Pi$, where $\pi \circ \sigma \in \Pi$ is defined as $(\pi \circ \sigma)(B) = \pi(\sigma(B)), \forall B \in \mathcal{B}$;

Faithfulness: if $f(\pi_B) = B$ for all $B \in \mathcal{B}$;

Consistency: if whenever $f(\pi) \cap f(\pi') \neq \emptyset$ for $\pi, \pi' \in \Pi$, we have $f(\pi + \pi') = f(\pi) \cap f(\pi')$;

Cancellation: if whenever $\pi \in \Pi$ satisfies $v(x, \pi) = v(y, \pi)$ for all $x, y \in X$, then $f(\pi) = X$.

For interpretations of the axioms, see Fishburn (1978a,b) and Xu (2010). We now state and prove our main result (of which part **(ii)** is already established in Fishburn 1978a):

Theorem 1. *Let f be a BAF. Then*

(i) *f satisfies neutrality, consistency and cancellation if and only if it is either a trivial function, or an approval voting, or an inverse approval voting, and*

(ii) *in addition, such f is faithful if and only if it is an approval voting.*

Proof. Since the IF parts are easy to prove, we prove the ONLY IF parts.

(i) We proceed in 4 steps. We remark here that Steps $2,3$ in our proof are the same as Steps $1,2$ in the proof of Theorem 1 in Alós-Ferrer (2006).

Step 1: Let us prove that for all $B \subseteq X$, $f(\pi_B)$ is either B, or $X \backslash B$ or X, and similarly, $f(\sum_{x \in B} \pi_x)$ is either B, or $X \backslash B$ or X. To see this, note that both $\pi_B \in \Pi$ and $\sum_{x \in B} \pi_x \in \Pi$ are invariant under any permutation $\sigma_B \in S$ that permutes the elements of B and $X \backslash B$ in an arbitrary way, but does not interchange the elements of these two sets. That is, for any such $\sigma_B \in S$,

$$\pi_B \circ \sigma_B = \pi_B$$

and

$$\Big(\sum_{x \in B} \pi_x\Big) \circ \sigma_B = \sum_{x \in B} \pi_x.$$

Then, by neutrality so must be $f(\pi_B)$ and $f(\sum\limits_{x \in B} \pi_x)$: for any such $\sigma_B \in S$,

$$\sigma_B(f(\pi_B)) = f(\pi_B)$$

and

$$\sigma_B\Big(f(\sum_{x \in B} \pi_x)\Big) = f(\sum_{x \in B} \pi_x).$$

But it is easy to check that the only sets in $2^X \backslash \{\emptyset\}$ with such property (invariant under any $\sigma_B \in S$) are B, $X \backslash B$ and X.

Step 2: Let us prove that for all $\pi \in \Pi$ and for all B', $B'' \subseteq X$ such that $B' \cap B'' = \emptyset$, we have

$$f(\pi + \pi_{B' \cup B''}) = f(\pi + \pi_{B'} + \pi_{B''}).$$

To see this, note that cancellation implies that

$$f(\pi_{B'} + \pi_{B''} + \pi_{X \backslash (B' \cup B'')}) = X.$$

Then, by consistency,

$$f(\pi + \pi_{B' \cup B''}) = f(\pi + \pi_{B' \cup B''} + \pi_{B'} + \pi_{B''} + \pi_{X \backslash (B' \cup B'')}).$$

Note also that by cancellation,

$$f(\pi_{B' \cup B''} + \pi_{X \backslash (B' \cup B'')}) = X$$

and then by consistency,

$$f(\pi + \pi_{B'} + \pi_{B''}) = f(\pi + \pi_{B'} + \pi_{B''} + \pi_{B' \cup B''} + \pi_{X \backslash (B' \cup B'')}).$$

Hence,

$$f(\pi + \pi_{B' \cup B''}) = f(\pi + \pi_{B'} + \pi_{B''}). \tag{1}$$

Step 3: Let $\pi \in \Pi$ be an arbitrary voter response profile. Let $\pi' \in \Pi$ be such that $v(x,\pi) = v(x,\pi')$ for all $x \in X$, but π' consists only of singleton ballots, i.e. $\pi'(B) > 0$ implies that $|B| = 1$. The profile π' is constructed from π by taking apart each ballot cast under π into separate, singleton ballots. Then iteration of (1), starting from $\pi^0 \in \Pi$, shows that $f(\pi) = f(\pi')$.

Step 4: Let $x \in X$ be any alternative and consider $\pi_x \in \Pi$. By Step 1, $f(\pi_x)$ is either $\{x\}$, or $X\backslash\{x\}$, or X. We show that these possibilities correspond, respectively, to the cases of f being either f^A, or f^{-A}, or f^*.

Case 1: Let $f(\pi_x) = \{x\}$. By neutrality, $f(\pi_y) = \{y\}$ for any $y \in X$. We claim that for all $B \subseteq X$, $f(\pi_B) = B$, i.e. f is faithful. Since by cancellation (or by neutrality), $f(\pi_X) = X$, we may assume that $B \subsetneq X$. Note that by Step 1, $f(\pi_B)$ is either B, or $X\backslash B$, or X. Suppose $f(\pi_B) \neq B$, then consistency implies that for $z \in X\backslash B$,

$$f(\pi_B + \pi_z) = f(\pi_B) \cap f(\pi_z) = \{z\}.$$

By (1),

$$f(\pi_B + \pi_z) = f(\pi_{B\cup\{z\}}),$$

which implies that $f(\pi_{B\cup\{z\}}) = \{z\}$. But that contradicts to Step 1. So, $f(\pi_B) = B$ and the claim is established. Then, by repeating the same argument as in Step 3 of the proof of Theorem 1 in Alós-Ferrer (2006), we can conclude that $f = f^A$.

Case 2: Let $f(\pi_x) = X\backslash\{x\}$. By neutrality, $f(\pi_y) = X\backslash\{y\}$ for all $y \in X$. We claim that for all $B \subsetneq X$, $f(\pi_B) = X\backslash B$. Note that by (1),

$$f(\pi_B) = f(\sum_{z\in B} \pi_z),$$

and starting from any two elements, $z_1, z_2 \in B$, by repeated use of consistency,

$$f(\sum_{z\in B} \pi_z) = \bigcap_{z\in B} X\backslash\{z\} = X\backslash B$$

which implies that, $f(\pi_B) = X\backslash B$ and the claim is established. Let $\pi \in \Pi$ be a given profile. Let $K = \max\{v(x,\pi)\}$ and note that K is well defined since X is finite. For each $k = 0,\ldots,K$, we define $B_k = \{x \in X : v(x,\pi) = k\}$. Then, the sets B_k form a partition of X. Consider the profile

$$\pi^* = \pi_{B_K} + \pi_{B_K\cup B_{K-1}} + \ldots + \pi_{B_K\cup B_{K-1}\cup\ldots\cup B_1}.$$

Since for $B \subsetneq X$, $f(\pi_B) = X\backslash B$ and $f(\pi_X) = X$, consistency implies that

$$f(\pi^*) = X\backslash(B_K \cup B_{K-1} \cup \ldots \cup B_{j+1}) = B_j,$$

where $j = \min\{k : B_k \neq \emptyset\}$.[1] But iteration of (1) implies that $f(\pi^*) = f(\pi')$ and

[1] Notice that by construction, $B_K \cup B_{K-1} \cup \ldots \cup B_j = X$, and hence for $0 \leq i \leq j$, $f(\pi_{B_K\cup\ldots\cup B_i}) = X$.

by Step **3**, we conclude that $f(\pi) = B_j$. Thus, $f = f^{-A}$.

Case 3: Let $f(\pi_x) = X$. Then by neutrality, $f(\pi_y) = X$ for all $y \in X$. We claim that for all $B \subseteq X$, $f(\pi_B) = X$. Since $f(\pi_X) = X$ by cancellation (or by neutrality), we can assume that $B \subsetneq X$. By consistency,

$$f(\pi_B + \sum_{z \in X \setminus B} \pi_z) = f(\pi_B) \cap X = f(\pi_B)$$

since $\forall z \in X \setminus B$, $f(\pi_z) = X$. But by cancellation,

$$f(\pi_B + \sum_{z \in X \setminus B} \pi_z) = X.$$

Hence, $f(\pi_B) = X$ and the claim is established. As in Case 2, let us consider $\pi^* \in \Pi$. Since $f(\pi_{B_k}) = X$ for $0 \leq k \leq K$, consistency implies that $f(\pi^*) = X$. But iteration of (1) implies that $f(\pi^*) = f(\pi')$ and by Step 3, we conclude that $f(\pi) = X$. Thus, $f = f^*$.

(ii) Clearly, none of f^{-A} and f^* is faithful. Hence, $f = f^A$. □

3. Final remarks

From the outset, it may seem that the primary role of the axiom of faithfulness is to fix an orientation, i.e. to set the right direction. The analysis above clarifies that intuition: in the axiomatization of Approval Voting (AV) in Fishburn (1978a), faithfulness helps us to distinguish AV from a function that always chooses the whole set of alternatives, and a function that always chooses the least approved alternatives.

Finally, there is an interesting similarity between Theorem 1 and Wilson's impossibility theorem, which is obtained as a consequence of dropping the Pareto axiom in Arrow's impossibility theorem. It states that, a social welfare functions satisfies non-imposition and independence axioms if and only if it is either a trivial function (null function), or a dictatorial function, or inversely dictatorial function (see Wilson 1972).

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