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Pollution Control Policy: A Dynamic Taxation Scheme

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Abstract In this paper we investigate a dynamic setting of environmental taxation, for which the government imposes a tax rate in order to internalize externalities caused by polluting firms. The basic model consists of the intertemporal maximization problem for an additively separable utility which is subject to the pollution accumulation constraint. We analyze some various aspects of the same setting such as the leader-follower, the social planning and the simultaneous move game. The model is very simple and has some similarities with capital taxation models. The crucial variables of the model are the tax rate as a control and the pollution stock as a state. We discuss a scheme in which polluters adopt Markovian emission strategies with respect to the flows, and the tax rate imposed is, in the most cases, a constant depending on the discount rate. Moreover, computing the time paths for the control and state variables, the welfare index analysis, that follows, reveals substantial inefficiencies caused by the leader-follower setting, compared with the social planning optimal control setting.

Keywords Pollution control, taxation, dynamic leader-follower games
JEL classification C61, C62, D43, H21

1. Introduction

Pollution control, associated with production process, has already become a field of intensive economic study during the last decades. There have been various methods suggested so that the externalities caused by the flow of pollutants can be curtailed. These methods include the taxation of firms according to the flow of pollutants they create during the production process, the enforcement of pollution standards, the government demands which force the firms to buy pollution permits and so on.

Concerning taxation, there are a lot of papers in recent literature (e.g. Benchekroun and Long 1998) which address the problem through the following dynamic model: the announced by the government tax rule is an exogenized functional, and the due tax payments for the pollution caused in time t is a function of the quantity of the output produced at the same time t in association with the pollution level. The Benchekroun-Long example focuses on stationary Markovian tax rules which are linear as to the produced output, but not necessarily linear as to the pollution stock.

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Long and Soubeyran (2005) demonstrate that the tax rates per pollutants unit are not identical for all the producers who operate in a competitive market, and call their finding *property of selective penalization*. Moreover, they prove the "Optimal Distortion Theorem", showing that an efficient tax structure demands that high cost firms pay a higher tax rate. They analyze the optimal tax structures of penalties for polluting firms with heterogenous costs and show that there is a bias in favor of efficient firms, so, as far as efficiency is concerned, a structure of systematic biases emerges. Finally, their analysis can be used to study the role of strategic trade policy in the presence of a polluting international oligopoly.

Our paper will be based on the dynamic version of the leader-follower Stackelberg model, one of the most powerful tools which are able to analyze the economic interactions among firms—or even countries—with economic action. In its original form, the Stackelberg model requires a minimum of two economically active actors, the most powerful one being the leader who announces its strategy, while the rest of the actors (the followers) respond or react to the leader's announced strategies. The same model can be applied on transactions between countries and the agents of economic activity inside these countries, in which case the role of the leader is assumed by the announced government policy. Specifically, we propose a Stacklberg dynamic model for environmental pollution taxation, in which the government policy imposes a tax rule $\tau(t)$ at the start of the leader-follower games in order to regulate the externalities which stem from polluting firms. The firms which are involved in a country's market have the choice to pay lower taxes, on condition that they invest on effective, pollution reducing technologies. The motive given by government policy to polluting firms so that they may invest on pollution mitigation technology is tax exemption for this particular investment. Our analysis might prove particularly useful for a country which is forced to comply with environmental agreements, such as the Kyoto protocol.

Finally, the main contribution of our paper is the fact that the environmental problem in the form of pollutants stock externality is faced as a conflict, i.e. as a dynamic game between the government and the polluting firms. This consideration completes the existing literature which faces the same environmental problem as an optimal control one, which implies the absence of strategically acting agents. At the same time our results extend those of Farzin (1996) as our proposed model compares several game theoretic settings making use of the welfare index methodology. Additionally, another significant contribution of the suggested model is that the tax rate is entered into the constraint of the maximization problem instead of the objective function. In this way, our task is to explore optimal tax schemes and not to maintain the basic result of Pigou, the well known Pigovian taxation.

The structure of the paper is as follows. Section 2 reviews the existing relative literature and comments on linear Markov strategies. Section 3 describes the proposed model, while section 4 analyses the equilibrium for the leader-follower game. Section 5 examines equilibria of the model without the government's intervention as leader. Section 6 constructs a simplified welfare performance index and finally Section 7 concludes the paper.

2. Literature review

In leader-follower games, where the government policy acts as the leader trying to influence the economically active agents (i.e. the followers of the announced government policy) the government's optimal program usually suffers from the phenomenon of time inconsistency. The concept of *time inconsistency of optimal policy* is extremely significant for modern economic theories. Since the influential work by Kydland and Prescott (1977), economists have attempted different approaches to resolve the inconsistency problem. One possible strategy is to consider the interaction between the policy maker and the agent in a dynamic game setup.

Many researchers such as Cohen and Michel (1998) found that a time consistent outcome corresponds to a feedback Nash equilibrium, while the open loop Stackelberg equilibrium corresponds to a time inconsistent policy. Time consistency of the tax policy has an intuitive sense: the effect of the tax on the agent's present discounted value of future utility must be independent of the pollution level. It is widely known that time inconsistency hinges on the controllability of the follower's co-state variable in differential Stackelberg games with open loop informational structure. Here, controllability of the follower's co-state variable is defined as to whether this variable is independent of the leader's control path. Dockner et al. (2000) show that if the follower's co-state variable is uncontrollable by the leader, then the open-loop Stackelberg equilibrium is time consistent. However, time consistency is highly dependent on the game's special structure. Things seem to improve when the players of the game use Markovian strategies. Since Markovian strategies are time consistent by default, the result of the equilibrium is expected to be time consistent with a Stackelberg game due to the use of these strategies.

The existing literature on Markovian strategies is limited to examples of optimal capital tax and income redistribution. Optimal capital taxation based on time consistent Markovian strategies has been systematically examined by Kemp et al. (1993) in continuous time and by Krusell (2002) in discrete time. By taking the Markovian strategies into account, they claim that the optimal equilibrium tax rate is non-zero in general, because the social planner is allowed to deviate from zero tax policy without any commitment on behalf of the government. Strulik (2003) makes a valuable addition to the literature of optimal capital taxation and redistribution by discussing adjustment dynamics.

In the particular case of logarithmic utility, Strulik (2003) offers us a qualitative analysis as well as some necessary numerical calculations of the adjustment paths for the case of isoelastic utility. Lindner and Strulik (2004) analyze the Markovian Stackelberg strategy, based on the median voter static model by Alesina and Rodrik (1994). Within the constant marginal returns of capital they find that the optimal tax rates trajectories are time independent, so the Stackelberg solution is time consistent and, therefore, adjustment dynamics exist neither in economic growth nor in the growth of the public sector.

Some models which examine the problem of time inconsistency are, among others, those of Xie (1997) and Karp and Ho Lee (2003). By using a dynamic model of output taxation with capital accumulation, Xie (1997) proves that if a boundary (transversality) condition is necessary for optimality, then the government policy is time inconsistent, as the result is a zero tax policy which cannot be optimal. Nevertheless, by imposing two further boundary conditions, Xie shows that the original boundary condition is spurious for optimality. Karp and Ho Lee (2003) generalize Xie's findings through the use of the same model. Assuming that the government tax policy is bound to be multiplicatively separable, it is possible to establish the tax base function $b(t)$, which can provide a time consistent tax given by the expression $b(t)\,\tau(t)$.

Concerning Pigovian taxation, Farzin (1996) proposes a model showing, in the linear case, that the optimal tax policy consists of an environmental tax that increases at the discount rate over the period before the pollution stock attains its threshold level. Then it remains constant coupled with a constant resource depletion tax. Policy implications derived from Farzin's model, are expressed as percentage of welfare loses for the model's several formulations. Specifically, replacing the optimal tax with a constant tax rate implies less than 1.4 % welfare loss, compared with the optimal tax. At the same time a no-tax policy that fails to correct stock externalities, induces an estimated welfare loss of as much as 30 %.

The main connection of our paper and therefore a contribution to the existing literature could be the extension of Farzin's (1996) model. As he concludes, his model can be extended by explicitly allowing investments in new abatement technologies. Moreover, among the questions raised, as extensions of the Farzin's model, are what will the characteristics of the optimal investment policy be and how will they affect the optimal tax policy. The straightforward answer given by our proposed model is the basic adoption of pollutants accumulation. That is, once the abatement process is left on producers' side, the existing policy is given by the motive of tax exemption on investment. With the government acting as the leaded and announcing the tax rate, the leader-follower setting yields inefficiency with respect to social welfare. At the same time the social planning setting is more efficient compared to the Stackelberg one which in turn is more efficient than the Nash setting.

In our suggested environmental model, the firms which produce the output accumulate pollutants, but every polluting producer is motivated to invest on technologies which mitigate emissions. These technologies improve the quality of our environment, thus promoting the social welfare. Moreover, the government policy promotes decontamination by using the tax revenues raised from the taxation of polluting firms. What is more, in this model we consider pollution to be an undeniable fact which occurs during the production process and can only be moderated through the investment on cleaner technologies. For this reason, we introduce the tax rule into the equation of pollutants motion in order to investigate optimal tax schemes, calculate, and comment on optimal time paths of both the investment in pollution abatement technologies and the tax rule.

2.1 Linear Markov strategies

Typically, the Markov Perfect (MP) strategies are decision rules for the players of a dynamic game, which result to efficient decision taking. Markov strategy spaces for differential games, especially in the case of pollution control games, are defined as:

$\varphi_{MP} = \{e(s(t),t) \,|\, e(s(t),t)$ is Lipschitz-continuous w.r.t. $s(t)$ and continuous w.r.t. $t\}$, where $e(s(t))$ symbolizes the emission of pollutants as a function of the flow of pollutants.

The Markov Perfect equilibrium is the equilibrium during which the players of the game make use of Markovian strategies. Moreover, the Markov Perfect equilibrium has the strong property of subgame perfectness.

Dockner et al. (2000) explain that there are two types of Markovian strategies, the linear and non-linear ones. The linear Markovian strategies are particularly helpful, as they simplify calculations and are also global strategies, i.e. they can be defined for all the possible rates of the pollution stock.

Because of the linear Markov strategies, the players of a game have to accept decision rules which are connected with the pollutants flow in a linear way, that is in our case $e(s(t)) = Ds(t)$, where $e(s)$ is the emission of pollutants as a function of pollutants flow, while D is a positive constant which measures how efficient is technology that used by the production process.

Dockner and Long (1993) show that, in a differential game of pollution control between two countries, the equilibrium strategies can be linear or not. In case the equilibrium strategies are linear, they can be represented in the form of $e(s(t)) = a - bs(t)$ where a,b are positive constants. The previous Markov strategies differ from the linear version we accept as to the sign of the constant D. The non-cooperative game theoretic explanation given by Dockner and Long (1993) is: when the pollution stock is large, every player of the game will have some incentive to reduce its emission rate. In a simultaneous (Nash) game the incentive to reduce emissions is interpreted as follows.

A linear Markov decision rule implies that an increase (decrease) in the stock $s(t)$ leads to the decrease (increase) of optimal emissions. Suppose that player 1, in a Nash setting, finds it optimal to reduce its emissions. This causes the level of the pollution stock to decline. Since a clean environment is a public good and the other player(s) of the game benefit from this decline in pollution, they decide to emit more, reacting according to linear Markovian decision rules. In the long run, this behavior results in a higher steady state pollution stock. In our setting we presuppose that the incentive to reduce emissions rate is given by the adoption of the pollutants abatement investment decision.

Our paper is probably close, with respect to the adoption of emissions function, with a paper of Xepapadeas (1992). Xepapadeas (1992) distinguishes firm's emissions in discharges into environment and firm's net emissions, but the final result of emissions is clearly a function of pollutants flows (which is called by the author discharges) and abatement in the following convex fashion $e(S,A) = S(k_p, l_p) - A(k_a, l_a)$, e are emissions, S the pollutants flows, A the undertaken by firms abatement, k, l denotes the inputs capital and labor respectively, while the subscripts p, a used in capital and labor are the productive and abatement inputs.

3. The proposed model

Suppose there is a number N of producers in a market. The producers manufacture a simple product which is consumed, thus preventing its stock up. The production process creates an emission of pollutants which is subject to a tax rule that is enforced by the government. The tax rule is announced by the government at time zero of the game, so we set $\tau(t)$ as the tax rule and also set $\{\tau(t) \in [0,1], t \geq 0\}$. We also usually assume the premise that the revenue raised by the polluters' taxation is used as government expenditure for the provision of the public good of decontamination. Decontamination can be seen as a way of clearing solid, liquid and gaseous pollutants, and revolves around the concept of regenerating the environment and restoring it to its previous condition.

We will now try to outline the way in which there can be a connection between the output flow, which is symbolized as $q(t)$, and the flow of emissions during the production process, which is symbolized as $s(t)$. In the relevant literature, the equation which describes the rate of change of pollutants is, basically, a function of the produced output minus the ability of nature to decontaminate itself. Due to the lack of internal feedback, the most common equation which describes the accumulation of pollutants is $ds/dt = \dot{s} = q(t) - \delta s$, where $s(t)$ is the pollutants flow at time instant t, $q(t)$ is the output produced at the same time, while δ is the natural decay rate. Internal pollutants feedback appears in closed ecosystems, e.g. shallow lakes. For a superb study on the phenomenon of internal feedback see Mäler et al. (2003) and Kossioris et al. (2009).

Indeed, the aforementioned equation is in force on condition that the firms of a competitive or oligopolistic market do not relate the output level to the flow of emissions. However, if there is an obligation on behalf of the firms that their output level must depend on the level of emissions, then the produced output can be expressed as a function of pollution flows, i.e. $q = \chi(s)$. The last expression is known as Markovian representation, because the firms (the players of a Nash or Stackelberg game) employ Markovian strategies.

In our proposed model, the equation of the development of pollutants is derived under certain conditions. Firstly, we make the assumption that the output level of the producers involved in the economic environment is contingent on the quantity of pollutants that they emit, i.e. these producers employ Markovian strategies. Secondly, we assume that the firms involved are aware of the fact that the emission of pollutants during the production process harms the environment, thus reducing the social prosperity. The emission of pollutants is detrimental not only to the society but also to the producers themselves, as it is much harder for a polluting market to attract buyers. For this reason, the producers have some motives for investment on cleaner technologies which reduce the emission of pollutants, but these investments are costly.[1] Thirdly, the social planner, which usually is the government policy, enforces a tax on the firms only for the produced pollutants flow, exempting the investments on cleaner production tech-

[1] Effects on investment in abatement process, relate mainly to investment in pollution abatement equipment which prevent pollutants diffusion. This stock of abatement capital, in contrast to productive capital, can, therefore, regarded as a defensive expenditure which might be differentiated from productive capital by differences in installation, training costs, etc.

nologies from taxation. In this way, firms are encouraged to produce a higher quantity of clean output.

But a critical issue arises. If the tax revenue raised by the government is to be used for the provision of the public good of decontamination, which is the point where the polluting producer's liability stops and the social planner's (i.e. the government policy's) responsibility starts? It seems reasonable to assume that, since the polluting firm fulfills its tax obligation for the pollution it causes, then the responsibility shifts to the government policy which promotes the public good of decontamination. Therefore, after these simplifying assumptions, we can conclude that the equation of pollutant accumulation will be the difference of the two flows, i.e. the remainder of the pollutant flows after taxation minus the flow of investment on cleaner technologies which aim at the mitigation of emissions.

Following the aforementioned hypotheses, the form of the equation which describes the pollutants flow rate of change, i.e. their accumulation, is the following:

$$\dot{s}_i(t) = \frac{ds_i}{dt} = e(s_i(t)) - \tau(t)\, e(s_i(t)) - A_i(t) = [1-\tau(t)]\, e(s_i(t)) - A_i(t),$$

where $A_i(t)$ is the cost of investment on pollution abatement technologies from producer i, $e(s_i(t))$ is the function of pollutants emission with pollutants flow $s_i(t)$, while $G(e(s_i(t)), \tau(t)) = \tau(t)\, e(s_i(t))$ is the payment function of the polluter, i.e. the function of tax revenue.

As already mentioned, the tax revenue is used by the government policy for the promotion of the public good of environmental renewal and restoration, and it is well-known that public goods bring about social utility. Investment on cleaner technologies from the producer side also brings about social utility. These two utilities can be expressed as $V(G)$ and $U(A_i)$, respectively. Furthermore, we suppose that utilities $V(G)$ and $U(A_i)$ are concave and increasing functions with respect to their arguments. We also set $U'(0) = V'(0) = +\infty$, and impose condition $A_i(0) = 0$ to prevent tax avoidance at the start of the game.

The problem which is presented below is a differential game with a leader-follower structure (Stackelberg game), during which the central planner who announces the tax rule at time zero of the game acts as the leader, whereas the representative producer reacts as a follower who determines a strategy having taken into account the leader's announcement.

Consequently, the problem of maximization of the representative producer is shown as follows:

$$\begin{aligned} &\max \int_0^\infty e^{-\rho t}[U(A_i(t)) + V(G)]dt \\ \text{s. t.} \quad &\dot{s}_i(t) = [1-\tau(t)]\, e(s_i(t)) - A_i(t) \end{aligned}$$

and, assuming that all the polluting producers are identical, we can remove the subscripts .Thus, the problem becomes as follows (the time variable is neglected for simplification purposes without any loss of notation):

$$\max_{A} \int_{0}^{\infty} e^{-\rho t} \left[U(A) + V(G) \right] dt \tag{1}$$
$$\text{s.t.} \quad \dot{s} = [1-\tau] e(s) - A$$

4. Equilibrium analysis

4.1 The leader-follower equilibrium

We examine the open loop equilibrium and solve the problem starting with the problem of the follower's maximization. Formulating the follower's Hamiltonian we get

$$H_F(G,A,\mu,t) = U(A) + V(G) + \mu_F([1-\tau(t)]e(s) - A). \tag{2}$$

The necessary first order conditions are:

$$\frac{\partial H_F}{\partial A} = 0 \Leftrightarrow U'(A) - \mu_F(t) = 0 \tag{3}$$

and

$$\dot{\mu}_F(t) = -\frac{\partial H_F}{\partial s} + \rho \mu_F(t) \Leftrightarrow \dot{\mu}_F(t) = (\rho - [1-\tau(t)])\mu_F(t). \tag{4}$$

Solving the first condition (3) results in the investment decision variable $A_F(t)$[2] as a function of costate variable $\mu(t)$, i.e. $A_F(t) = f(\mu(t))$, where $f(\mu_F) = (U')^{(-1)}(\mu_F)$ and $(U')^{(-1)}$ is the inverse function of the first derivative of the utility function $U'(A)$. Then, applying the implicit function theorem we can see that $f'(\mu_F) = -1/U''(f(\mu_F))$ > 0. The last relation entails that, if we can find functions $\mu_F(\cdot)$ and $s(\cdot)$ which serve the following differential equations

$$\dot{s}(t) = f(\mu_F(t))[1-\tau(t)]e(s(t)), \tag{5}$$

$$\dot{\mu}_F(t) = (\rho - [1-\tau(t)])\mu_F(t), \tag{6}$$

and the boundary conditions $s(0) = s_0$ and

$$\lim_{t\to\infty} \mu_F(t)s(t) = 0, \tag{7}$$

then the follower's optimal open loop strategy will be expressed through the function $A(\cdot) = f(\mu_F(t))$. Equations (5)–(7) characterize the follower's best response to the leader's control path.

We may now return to the problem of the leader's optimal control. The social planner, who acts as the leader, decides on a tax rate $\tau(t)$. The tax revenue used for the provision of government services is represented as $G(t) = \tau(t)e(s(t))$. The leader is already aware of the followers' best response to each of his control paths $\tau(t)$. His

[2] $A_F(t)$ is the follower's instant private investment determination.

problem of optimization lies in finding the control path $\tau(t)$ which maximizes the integral of discounted net benefit (or utility). The leader's utility is additively separable and consists of, first, the utility produced by the investment decision from the producers' side, and second, the utility derived from the government services. The leader's maximization problem that emerges is subject to both constraints: the pollution accumulation constraint and to the follower's best response constraint, which is known to the leader.

Consequently, the government's problem is formulated as follows:

$$\max \int_0^\infty e^{-\rho t}\left[U(A)+V(G)\right]dt$$

$$\text{s.t.}\quad \dot{s}(t)=[1-\tau(t)]\,e(s(t))-f(\mu_F(t))$$

$$\dot{\mu}_F(t)=(\rho-[1-\tau(t)])\,\mu_F(t)$$

The leader's problem treats s, μ_F as state variables and the tax rate, τ, as a control variable. Then the leader's Hamiltonian is formulated as

$$H_L=U(A)+V(G)+\psi_L\left[[1-\tau(t)]\,e(s(t))-f(\mu_F(t))\right]+\xi_L\left[(\rho-[1-\tau(t)])\,\mu_F(t)\right],$$

where ψ_L,ξ_L are the co-state variables of the states s,μ_F respectively. The F.O.C.s for optimality are:

$$\frac{\partial H_L}{\partial \tau(t)}=0$$

$$\dot{\psi}_L(t)=-\frac{\partial H_L}{\partial s}+\rho\psi_L=\psi_L\left[(1-\tau)+\rho\right]$$

$$\dot{\xi}_L(t)=-\frac{\partial H_L}{\partial \mu_F}+\rho\xi_L=-\psi_L f'(\mu_F(t))-\xi_L(\rho-(1-\tau))+\rho\xi_L$$

From the above formulation, time inconsistency can be seen as follows. Supposing we have solved the problem for the leader, then the function $\psi_L(\cdot)$ determines the time path of the leader's announced tax rate at time zero of the game. Let the game proceed and the leader adheres to his announcement. Then at some time instant $t_1>0$ the state and co-state variables have some values ψ_L^*, $s^*(t_1)$, $\mu_F^*(t_1)$, ξ_L^* with $\xi_L^*(t_1)\neq 0$. Suppose that at time instant t_1 the leader deviates from the announced time path. At the new time starting point the leader imposes the new tax rule $\tau(t_1)=\tau^*(t_1)$, which must be considered as given. However, it does not have to take the $\mu_F(t_1)=\mu_F^*(t_1)$ as a given initial condition but chooses a new initial condition $\mu_F(t_1)\neq\mu_F^*(t_1)$. Therefore, the optimal value of the associated co–state variable is $\xi_L=0$, but this implies that the new solution is not a continuation of the original problem solution.

Then we elaborate on those utility functions which exhibit constant elasticity of intertemporal substitution and are generally used in models of economic growth. These functions are in the following form

$$F(u)=\begin{cases}\dfrac{u^\beta-1}{\beta} & \text{if}\quad \beta\in(0,1)\\ \ln(u) & \text{if}\quad \beta=0\end{cases}$$

and the elasticity of intertemporal substitution is given by the expression $1/(1-\beta)$. For the special case for which $\beta = 0$ we have $F(u) = \ln(u)$ and elasticity equals to one.

In the case of this specification, especially for logarithmic utility, we have the following proposition.

Proposition 1. . *The proposed environmental leader-follower model yields time consistent results in the case of the special form of utilities. The tax rate imposed by the government on the polluting firms has a constant value independent of pollutants flows and time, given by the expression $\tau = \rho/2D$.*

Proof. See Appendix.

We find that the tax rule $\tau(t)$ is independent of the pollution flows $s(t)$. This fact gives raise that the game is not only time consistent but also sub-game perfect. We are able now to find the time paths expressions of the pollutants flows, firm's investment in abatement technologies and tax revenues as functions of any initial pollution stock s_0.

Corollary 1. *The trajectories of pollutants flows and investment in emission reducing technologies are given by the expressions $s(t) = s_0 e^{\left(\frac{2D-3\rho}{2}\right)t}$ and $A(t) = \rho s_0 e^{\left(\frac{2D-3\rho}{2}\right)t}$ respectively while the tax revenues for the decontamination public good are given by the expression $G(t) = \frac{\rho}{2} s_0 e^{\left(\frac{2D-3\rho}{2}\right)t}$. All the above functions are monotonically increasing functions of any initial pollution stock s_0 provided the discount rate and constant D, which measures the emissions of the pollutants flows, fulfill the inequality $D > \frac{3}{2}\rho$.*

Proof. Substituting the tax rate into the time paths of pollution flows and decontamination function respectively we have:

$$s(t) = s_0 e^{\int_0^t [(1-\tau(p))D-\rho]dp} = s_0 e^{\int_0^t \left(\frac{2D-3\rho}{2}\right)dp} = s_0 e^{\frac{2D-3\rho}{2}t} \tag{8}$$

$$A(t) = \rho s_0 e^{\int_0^t [(1-\tau(p))D-\rho]dp} = \rho s_0 e^{\int_0^t \left[\frac{2D-3\rho}{2}\right]ds} = \rho s_0 e^{\left[\frac{2D-3\rho}{2}\right]t} \tag{9}$$

Both (8) and (9) are monotonically increasing over time provided $D > 3\rho/2$, that is, for small discount rates.

The government services provision $G(t) = \tau(t)s(t)$ according to the tax rule and (8) is expressed as:

$$G(t) = \tau(t)Ds(t) = \frac{\rho}{2}s(t) = \frac{\rho}{2}s_0 e^{\int_0^t \left(\frac{2D-3\rho}{2}\right)dp} = \frac{\rho}{2}s_0 e^{\left(\frac{2D-3\rho}{2}\right)t}, \tag{10}$$

for which the increase requirement is the same as above, that is, for small discount rates, $D > 3\rho/2$. □

4.2 The model with N strategically acting followers

We consider the same original market setting, for which the market comprises the $N > 1$ firms, each of them chooses the abatement investment level $A_i(t)$, and behaves as a follower into the government's (leader's) tax policy setup, i.e. the tax rates $\tau_i(t)$. Every producer contributes a share according to the imposed tax rate $g_i(t) = \tau_i(t) D s_i(t)$ for the overall amount of the public good $G(t)$, but the whole public good is non excludable, that is $G(t) = \sum_{i=1}^{N} g_i(t) = \sum_{i=1}^{N} \tau_i(t) D s_i(t)$. After all, every polluting producer i faces the following problem:

$$\max_{A_i(t)} \int_0^{\infty} e^{-\rho t} \left[\ln A_i(t) + \ln G(t)\right] dt \tag{11}$$

$$\text{s.t.} \qquad G(t) = \sum_{k=1}^{N} \tau_k(t) D s_k(t) \tag{12}$$

$$\dot{s}_k(t) = (1 - \tau_k(t)) D s_k(t) - A_k(t) \tag{13}$$

$$s_k(0) = s_{k_0}, \quad k = 1, 2, \ldots, N \tag{14}$$

for which the corresponding current value Hamiltonian is:

$$H_{i,F} = \ln A_i(t) + \ln \left[\sum_{k=1}^{N} \tau_k(t) D s_k(t)\right] + \sum_{k=1}^{N} \lambda_{F_{ik}}(t) \left[(1 - \tau_k(t)) D s_k(t) - A_k(t)\right], \tag{15}$$

where $\lambda_{F_{ik}}$ is the co-state variable, evaluated at time t, that follower i associates with the state variable $s_j(t)$.

The necessary first order conditions for the problem (11)–(13) are the following:

$$-\frac{\partial H_{i,F}}{\partial A_i(t)} = \frac{1}{A_i(t)} - \lambda_{F_{ii}} = 0 \tag{16}$$

$$-\frac{\partial H_{i,F}}{\partial s_i(t)} = \dot{\lambda}_{F_{ii}}(t) - \rho \lambda_{F_{ii}}(t) \Leftrightarrow \dot{\lambda}_{F_{ii}}(t) = \lambda_{F_{ii}}(t) \left[\rho - D(1 - \tau_k(t))\right] \tag{17}$$

$$-\frac{\partial H_{i,F}}{\partial s_{ij}(t)} = \dot{\lambda}_{F_{ij}}(t) - \rho \lambda_{F_{ij}}(t) \Leftrightarrow \dot{\lambda}_{F_{ij}}(t) = \lambda_{F_{ij}}(t) \left[\rho - D(1 - \tau_k(t))\right], \forall j \neq i \tag{18}$$

Since the evolution of the $i's$ player state variable $s_i(t)$ is independent of the $j's$ state s_j and control A_j the game exhibits separate dynamics, and one can set $\lambda_{F_{ij}} = 0$, for every $i \neq j$, which means that the Hamiltonian of the i player can be written by taking into account the dynamics of the $i's$ state variable only.

Moreover, assuming symmetry across producers and across tax rates, i.e. $A_i(t) = A_j(t) = A(t)$, $s_i(t) = s_j(t) = s(t)$ and $\tau_i(t) = \tau_j(t) = \tau(t)$ for any i, j, t, we have $G(t) = N\tau(t) D s(t)$. As a consequence, we also have $\lambda_{F_{ii}}(t) = \lambda_{F_{jj}}(t) = \lambda_F$. The solution of the problem is $A(t) = 1/\lambda_F(t)$.

If we consider the government's position, the following problem arises:

$$\max_{\tau(t)} \int_0^\infty e^{-\rho t} N \left[\ln A(t) + \ln G(t)\right] dt \tag{19}$$

$$\begin{aligned} \text{s. t.} \qquad G(t) &= N\tau(t) Ds(t) \\ \dot{s}(t) &= D(1-\tau(t))s(t) - A(t) \\ s(0) &= s_0 > 0 \\ A(t) &= 1/\lambda_F(t) \\ \dot{\lambda}_F(t) &= \lambda_F(t)\left[\rho - D(1-\tau(t))\right] \end{aligned}$$

with the corresponding current value Hamiltonian

$$\begin{aligned} H_L \quad = \quad & N\ln(1/\lambda_F(t)) + N\ln(NDs(t)\rho(t)) + \\ & + N\lambda_L(t) D(1-\tau(t))s(t) - 1/\lambda_F(t) + \\ & + N\psi_L(t)\lambda_F(t)\left[\rho - D(1-\tau(t))\right]. \end{aligned} \tag{20}$$

The solution of the above problem is the same as in the original discussed in Section 4.1, and the trajectories of the complete solution are characterized by the following equations:

$$\begin{aligned} \tau &= \frac{\rho}{2D} \\ s(t) &= s_0 e^{\frac{2D-3\rho}{2}t} \\ A(t) &= \rho s_0 e^{\frac{2D-3\rho}{2}t} \\ G(t) = \tau(t) NDs(t) &= \frac{\rho}{2} N s_0 e^{(D-\rho)t} \end{aligned}$$

The outcome is time consistent, since the tax rate is constant.

In the case of the social planning with N identical producers, the optimal control problem is characterized by the following equations:

$$\max_{A(t),\tau(t)} \int_0^\infty e^{-\rho t} N \left[\ln A(t) + \ln(ND) + \ln s(t) + \ln \tau(t)\right] dt$$

$$\begin{aligned} \text{s. t.} \qquad \dot{s}(t) &= (1-\tau(t)) Ds(t) - A(t) \\ s(0) &= s_0 > 0 \end{aligned}$$

which is equivalent to the problem with one representative producer, and with solutions

$$\begin{aligned} \tau &= \frac{\rho}{2D} \\ s(t) &= s_0 e^{(D-\rho)t} \\ A(t) &= \rho s_0 e^{(D-\rho)t} \\ G(t) = \tau(t) NDs(t) &= \frac{\rho}{2} N s_0 e^{(D-\rho)t} \end{aligned}$$

with the same subgame perfectness property.

5. The model without the government's intervention (as leader)

5.1 The social planning optimal control problem

In this section we suppose that the benevolent social planner chooses both investment in abatement and the tax rate, using the information of the public good provision $G(t) = \tau(t) Ds(t)$. Then the optimal control problem consists of:

$$\max_{A(t),\tau(t)} \int_0^\infty e^{-\rho t} \left[\ln A(t) + \ln(Ds(t)\tau(t))\right] dt \tag{21}$$

$$\text{s. t.} \qquad \dot{s}(t) = (1-\tau(t)) Ds(t) - A(t) \tag{22}$$

The Hamiltonian current value H_{SP} is formulated as follows:[3]

$$H_{sp} = \ln A(t) + \ln(Ds(t)\tau(t)) + \lambda(t)\left[(1-\tau(t)) Ds(t) - A(t)\right] \tag{23}$$

The F.O.C.s are

$$\frac{\partial H_{sp}}{\partial A(t)} = 0 \Leftrightarrow \frac{1}{A(t)} = \lambda(t) \tag{24}$$

$$\frac{\partial H_{sp}}{\partial \tau(t)} = 0 \Leftrightarrow \frac{1}{\lambda(t) s(t)} = \tau(t) \tag{25}$$

$$-\frac{\partial H_{sp}}{\partial s(t)} + \rho\lambda(t) = \dot{\lambda}(t) \Rightarrow \dot{\lambda}(t) = \lambda(t)\left[\rho - D(1-\tau(t))\right] - \frac{1}{s(t)} \tag{26}$$

and the transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t) s(t) = 0. \tag{27}$$

Plugging (24) into the constraint (22) and multiplying both sides by $\lambda(t)$ we have

$$\dot{s}(t)\lambda(t) = \lambda(t)(1-\tau(t)) Ds(t) - 1. \tag{28}$$

Multiplying (26) by $s(t)$ results in

$$s(t)\dot{\lambda}(t) = s(t)\lambda(t)\left[\rho - D(1-\tau(t))\right] - 1. \tag{29}$$

Summing up (28), (29) the result is

$$\frac{d(\lambda(t) s(t))}{dt} = \rho s(t)\lambda(t) - 2 \tag{30}$$

with the solution $\lambda(t) s(t) = \frac{2}{\rho} + \Omega e^{\rho t}$.

In order to satisfy the transversality condition (27) we set the integration constant Ω equal to zero, so the solution turns to $\lambda(t) = 2/\rho s(t)$ from which, making use of (24), we end up to:

$$A(t) = \frac{\rho s(t)}{2}$$

[3] H_{SP} is the Hamiltonian current value for the social planer.

$$\tau = \frac{\rho}{2D}$$

The result is recorded in the next proposition.

Proposition 2. *The pollution control hierarchical game yields the same tax rate as in the benchmark case of social planning but different time paths of investment in emission reducing technologies.*

We are now able to compute all trajectories of the relevant variables for the social planner problem.

Corollary 2. *The trajectories of the investment in emission reducing technologies, pollution and tax revenues are given by the expressions* $A(t) = \frac{\rho}{2} s_0 e^{(D-\rho)t}$, $s(t) = s_0 e^{(D-\rho)t}$ *and* $G(t) = \frac{\rho}{2} s_0 e^{(D-\rho)t}$ *respectively. All the functions are monotonically increasing time functions, provided* $D > \rho$.

Proof. Substituting the tax rate into the pollutants evolution equation, we have

$$\dot{s}(t) = (1-\tau(t))Ds(t) - A(t) = \left(1 - \frac{\rho}{2D}\right)Ds(t) - \frac{\rho s(t)}{2} \Leftrightarrow \dot{s}(t) = (D-\rho)s(t)$$

with solution $s(t) = s_0 e^{(D-\rho)t}$ and the time path of the decontamination investment is after substitutions

$$A(t) = \frac{\rho}{2} s_0 e^{(D-\rho)t},$$

while the trajectory of the public good provision is

$$G(t) = \tau(t)Ds(t) = \frac{\rho}{2D}Ds(t) = \frac{\rho}{2} s_0 e^{(D-\rho)t}.$$

Notice that all trajectories are increasing functions provided $D > \rho$, that is, for small discount rates. $\square$

5.2 The simultaneous move game

In order to have a complete characterization of the model we discuss the case in which the government's intervention is absent, and the existing N firms in the market move simultaneously, choosing the time path of the abatement investment $A_i(t)$ and the time path share of the tax rate $\tau_i(t)$ which is multiplied by the emissions $e_i(s)$, thus giving the instant amount of the overall public good $G(t)$. Public good is non-excludable, so we can conclude that $G(t) = \sum_{i=1}^{N} \tau_i(t) D s_i(t)$.

Each atomistic firm i solves the following problem, with respect to the control variables $A_i(t), \tau_i(t)$:

$$\max \int_0^\infty e^{-\rho t} \left[\ln A_i(t) + \ln G(t)\right] dt$$

$$\begin{aligned} \text{s.t.} \qquad G(t) &= \sum_{k=1}^{N} \tau_k(t) D s_k(t) \\ \dot{s}_k(t) &= (1-\tau_k(t)) D s_k(t) - A_k(t) \\ s_k(0) &= s_{k_0}, \quad k = 1, 2, \ldots, N \end{aligned}$$

Solving the Nash game we found the same subgame perfectness outcome, but for different levels of investment in abatement technologies. We put the solution results in a proposition.

Proposition 3. *The Nash pollution control game, in which each producing firm chooses both investment in abatement technologies and its own share of the tax rate that contributes to the environmental decontamination public good, is subgame perfect and the investment decision result is dependent on the number of players acting in the market.*

Proof. See Appendix.

Corollary 3. *In the Nash setting, as the number of producers increases tending to infinity, the tax rate tends to zero, while the investment in abatement technology tends to the time path* $A(t,\infty) = \rho s_0 e^{(D-\rho)t}$.

Proof. In Appendix, we found the tax rate and abatement investment for the Nash game given by $\tau(N) = \frac{\rho}{D(N+1)}$, and $A(t,N) = \rho \frac{N}{N+1} s_0 e^{(D-\rho)t}$, then taking limits we have, $\lim_{N\to+\infty} \frac{\rho}{D(N+1)} = 0$ and $\lim_{N\to+\infty} A(t,N) = \rho s_0 e^{(D-\rho)t}$. □

6. The welfare performance index

We attempt to characterize the welfare given by the variations of the basic model. One measure of welfare is, obviously, the discounted value enjoyed by the society from the decontamination public good and from the private investment by the firms relative to the investment decision in cleaner technologies, which abates pollution. Thus, the welfare index could be

$$I_W = \int_0^\infty e^{-\rho t} \left[\ln A(t) + \ln G(t)\right] dt. \tag{31}$$

The comparison taking place is the result of the several types of structures discussed above in the previous sections. To be more precise, we compare the time paths found in the Stackelberg game, in the social planning setting and in the simultaneous move game.

In the leader-follower game the welfare index (31) takes the form

$$I_{W,\,Stackelberg} = \int_0^\infty e^{-\rho t} \left[\ln\left(\rho s_0 e^{\frac{2D-3\rho}{2}t}\right) + \ln\left(\frac{\rho N}{2} s_0 e^{\frac{2D-3\rho}{2}t}\right)\right] dt,$$

and after simplifications we finally have the index

$$I_{W,\,Stackelberg} = \frac{\ln(1/2) + \ln N + 2\ln(\rho s_0)}{\rho} + \frac{(2D-3\rho)}{\rho^2}.$$

In the social planning optimal control problem welfare index reduces to

$$I_{W,\,SP} = \int_0^\infty e^{-\rho t} \left[\ln\left(\frac{\rho s_0}{2} e^{(D-\rho)t}\right) + \ln\left(\frac{\rho N}{2} s_0 e^{(D-\rho)t}\right)\right] dt,$$

which after simplifications yields

$$I_{W,\,SP} = \frac{\ln(1/4) + \ln N + 2\ln(\rho s_0)}{\rho} + \frac{2(D-\rho)}{\rho^2}.$$

To that end, the Nash game index is

$$I_{W,\,Nash} = \int_0^\infty e^{-\rho t}\left[\ln\left(\frac{\rho N}{1+N} s_0 e^{(D-\rho)t}\right) + \ln\left(\frac{\rho N}{1+N} s_0 e^{(D-\rho)t}\right)\right] dt,$$

and the final simplified index is

$$I_{W,\,Nash} = \frac{2\ln[1/(N+1)] + 2\ln N + 2\ln(\rho s_0)}{\rho} + \frac{2(D-\rho)}{\rho^2}.$$

It is immediately obvious that $I_{W,\,SP} - I_{W,\,Stackelberg} = (1-\ln 2)/\rho > 0$, and $I_{W,\,SP} - I_{W,\,Nash} = [\ln((1+N)^2/4N)]/\rho$ which is greater than zero in the case $N > 1$ and zero for $N = 1$.

Finally, the difference of indices $I_{W,\,Stackelberg} - I_{W,\,Nash} = [\ln((1+N)^2/2N) - 1]/\rho$ is greater than zero in the $N > 1$ population case and negative for the population of polluting producers $N = 1$.

As it is widely known, that every Nash game collapses to an optimal control when the number of players reduces to one, and therefore leaves no room for policy making by the government. Thus, one can extract policy making conclusions only comparing the social planning optimal control and the leader-follower cases, when the population of the polluting producers is normalized to one. The simple welfare index construction reveals that the leader-follower game, with the government acts as a leader, yields inefficient outcome, compared with the social planning optimal control problem.

The same inefficient result is revealed by the index in the case of $N > 1$ acting polluting producers in a market, i.e., the welfare produced by the social planning is greater than in the Stackelberg setting. Additionally the leader-follower game yields higher welfare outcome than in the Nash setting, when the population is greater than one.

7. Conclusions

There is a wide area of pollution control policies available to regulators and each of them has different properties with respect to incentives for technological change. In this paper we focus on the basic instrument of taxing, available to the government, which is theoretically able to control pollution incurred by production process. A major assumption made in the present paper is that the firms acting in the entire market employ Markovian strategies, that is, they condition their production strategy according to the pollution flows that are emitted in the way $q = \chi(s)$.

Another assumption that is made in this paper is pollution's irreversibility and, therefore, market mechanism fails to correct pollution externality incurred by the producers acting in the entire market. Consequently, the government's intervention is needed in order for the existing economically acting firms in the market to comply with coupled production–pollution standards. For compliance purposes, a leader-follower setting (borrowed from the capital taxation field) is used by the proposed model. The leader, which is the government policy, announces a tax rate to the polluting firms and the followers, i.e. the producers, react choosing their policies.

Policies available to the firms are the investment decisions in abatement technologies that mitigate pollution. To be more specific, government policy gives the choice to economically producing agents in a market to pay taxes or to invest in cleaner technologies or both. It stands to reason that if the abatement investment expenditure is more in magnitude, then the pollution accumulation becomes less—but it obviously never vanishes. Therefore, the government always has to collect taxes for environmental regeneration, regardless of the investment decision on the firm's side. Tax revenue, as we mentioned, is used by the government for environmental decontamination in the form of a public good. According to the assumptions made, the value of the accumulated pollutants will be the value of the after tax emissions minus the value of the investment in abatement on the firm's side. The last assumption is the main hypothesis and contribution of the present paper.

Unfortunately, in the Stackelberg setting, the well known time inconsistency phenomenon arises, which is central in economic theory. Solving the dynamic model, first for the leader–follower setting, we find the subgame perfect tax rule which is a constant, therefore time consistent, and not dependent on emissions or on the number of the producers acting in the entire market. The social planning optimal control problem solution also reveals the same stationary tax rate, but different levels of the firms' investment decision. Finally, the simultaneous move game, for which each of N players of the game chooses both investment in abatement technologies and its own share of the tax rate contributing to the environmental decontamination public good, has solutions both for the tax rate and abatement decision, which depend on the numbers of the firms.

Having the results of the proposed model in a variety of information structures, we are in a position to construct the welfare index in order to conclude what is the most efficient solution. We construct the index as the integrand of the discounted overall utility, which consists of the utility enjoyed by the decontamination of the environment public good provision and of the firm's abatement expenditures in cleaner technologies that mitigate pollutants. Finally, the welfare index reveals a significant inefficiency when the government intervenes acting as a leader and announces the tax policy from which it may deviate. Among the informational structures employed in the model's solution, the most efficient one is the social planning structure, for which the benevolent planner chooses both abatement and the tax shares for each producer acting in the market, regardless the number of the firms. Obviously, the Stackelberg outcome for the proposed model is inefficient with respect to the social planning, where strategic interaction among government and polluting producers is absent by definition.

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Appendix

Proof of Proposition 1

The Hamiltonian current value, for the follower, is:

$$H_F = \ln(A) + \ln G(t) + \mu_F\left([1-\tau(t)]\,Ds(t) - A(t)\right)$$

The F.O.C. now yields

$$\frac{\partial H_F}{\partial A} = 0 \Leftrightarrow A(t) = \frac{1}{\mu_F(t)}. \tag{A1}$$

Equation $(A1)$ predicts that the follower's co-state variable $\mu_F(t)$ is independent of the leader's control variable $\tau(t)$ and therefore the follower's co-state variable is uncontrollable by the leader's control path. This predicts that the strategy is a time consistent one.

Substituting condition $(A1)$ into (1), in the main text, we have

$$\dot{s} = [1-\tau(t)]\,Ds(t) - \frac{1}{\mu_F(t)}. \tag{A2}$$

Multiplying both sides of $(A2)$ by $\mu_F(t)$ the last gives

$$\dot{s}(t)\,\mu_F(t) = D[1-\tau(t)]\,\mu_F(t)\,s(t) - 1. \tag{A3}$$

Setting the emissions function to be linear w.r.t the flows, i.e. $e(s(t)) = Ds(t)$, and multiplying both sides of the F.O.C. (4) by $s(t)$ gives

$$\dot{\mu}_F(t)\,s(t) = s(t)\,\mu_F(t)\left(\rho - D[1-\tau(t)]\right). \tag{A4}$$

Summing up $(A3)$ and $(A4)$ we have the differential equation

$$\frac{d\left(s(t)\,\mu_F(t)\right)}{dt} = \rho s(t)\,\mu_F(t) - 1. \tag{A5}$$

The solution of $(A5)$ is now

$$s(t)\,\mu_F(t) = \frac{1}{\rho} + \Omega_1 e^{\rho t}, \tag{A6}$$

where Ω_1 is the integration constant. In order to satisfy the tranversality condition (7) is necessary to set $\Omega_1 = 0$.

The solution of $(A6)$ turns to

$$s(t)\,\mu_F(t) = \frac{1}{\rho}. \tag{A7}$$

Combining equations $(A1)$ and $(A7)$ we get

$$A(t) = \rho s(t). \tag{A8}$$

Substituting $(A8)$ into (1), (in the main text)

$$\dot{s} = [1-\tau(t)]\,Ds(t) - \rho Ds(t), \tag{A9}$$

from which we get the solutions

$$s(t) = s_0 e^{\int_0^t [(1-\tau(s))D-\rho]ds} \text{ and } A(t) = \rho s_0 e^{\int_0^t [(1-\tau(s))D-\rho]ds}.$$

Now we can take the leader's position in order to build his Hamiltonian. The policy maker chooses the tax rule $\tau(t)$, so as to maximize the utility enjoyed by the investment determination and to maximize the discounted utility enjoyed by the public good provision, under the constraints represented as:

$$\dot{s}(t) = [1-\tau(t)]\,Ds(t) - A(t)$$

$$\dot{\mu}_F(t) = (\rho - [1-\tau(t)])\,\mu_F(t)$$

and under the follower's optimal investment decision which is given by

$$A(t) = \frac{1}{\mu_F(t)}.$$

In this way the leader's Hamiltonian current value for the problem becomes

$$\begin{aligned} H_L \quad = \quad & \ln\frac{1}{\mu_F} + \ln[D\tau(t)\,s(t)] + \psi_L\left[[D(1-\tau(t))]\,s(t) - \frac{1}{\mu_F}\right] + \\ & + \xi_L\mu_F\,[(\rho - [D(1-\tau(t))])], \end{aligned}$$

where ψ_L, ξ_L denote the co-state variables of the states s, μ_F, respectively. The F.O.C.s for the leader are

$$\frac{\partial H_L}{\partial\tau(t)} = \frac{1}{\tau(t)} - D(s(t)\,\psi_L(t) - \xi_L\mu_F) = 0 \tag{A10}$$

$$-\frac{\partial H_L}{\partial s(t)} = \dot{\psi}_L(t) - \rho\psi_L(t) \tag{A11}$$

$$-\frac{\partial H_L}{\partial\mu_F(t)} = \dot{\xi}_L(t) - \rho\xi_L(t) \tag{A12}$$

It is worth noting that, because of the feedback effects absence on either $(A11)$ or $(A12)$, the government's F.O.C.s are defined for an open loop solution.

Solving equation $(A10)$ with respect to the tax rate, gives

$$\tau(t) = \frac{1}{D[s(t)\,\psi_L(t) - \xi_L\mu_F]}. \tag{A13}$$

Equations $(A11)$ and $(A12)$ yield respectively

$$\dot{\psi}_L(t) = -\frac{\partial H_L}{\partial s(t)} + \rho\psi_L(t) = -\frac{1}{s(t)} + \psi_L(t)\,[\rho - D[1-\tau(t)]], \tag{A14}$$

$$\dot{\xi}_L(t) = -\frac{\partial H_L}{\partial \mu_F(t)} + \rho \xi_L(t) = \frac{\mu_F^2(t) D[1-\tau(t)]\xi_L(t) + \mu_F(t) - \psi_L(t)}{\mu_F^2(t)}. \tag{A15}$$

Substituting into the state dynamics the $\mu_F(t) = \frac{1}{\rho s(t)}$, we obtain

$$\dot{s} = [1-\tau(t)] Ds(t) - \rho s(t). \tag{A16}$$

Multiplying both sides of $(A16)$ by $\psi_L(t)$ yields

$$\begin{aligned} \dot{s}(t)\psi_L(t) &= \psi_L(t)[1-\tau(t)]Ds(t) - \psi_L(t)\rho s(t) \\ &= \psi_L(t)[1-\tau(t)]Ds(t) - \psi_L(t)A(t). \end{aligned} \tag{A17}$$

Multiplying both terms of $(A14)$ by $s(t)$ yields

$$\dot{\psi}_L(t)s(t) = s(t)\psi_L(t)[\rho - D[1-\tau(t)]] - 1. \tag{A18}$$

Summing up $(A17)$ and $(A18)$ we obtain

$$\frac{d[\psi_L(t)s(t)]}{dt} = -1. \tag{A19}$$

The differential equation $(A19)$ has the solution

$$\psi_L(t)s(t) = \Omega_2 - t. \tag{A20}$$

With Ω_2 to denote the integration constant. Combining $(A20)$ with $(A7)$ we have

$$\psi_L(t) = \frac{\Omega_2 - t}{s(t)} = (\Omega_2 - t)\mu_F(t)\rho. \tag{A21}$$

Relation $(A15)$ after substituting $(A21)$ and multiplying by $\mu_F(t)$ gives:

$$\dot{\xi}_L(t) = -\frac{\partial H_L}{\partial \mu_F(t)} + \rho \xi_L(t) = \frac{\mu_F^2(t) D[1-\tau(t)]\xi_L(t) + \mu_F(t) - \psi_L(t)}{\mu_F^2(t)}$$

$$\begin{aligned} \dot{\xi}_L(t)\mu_F(t) &= \frac{\mu_F^2(t)\xi_L(t)D[1-\tau(t)] + (\Omega_2 - t)\mu_F(t)\rho - \mu_F(t)}{\mu_F(t)} \\ &= \mu_F(t)\xi_L(t)D[1-\tau(t)] - (\Omega_2 - t)\rho + 1 \end{aligned} \tag{A22}$$

Now multiplying both sides of (4), in the main text, by $\xi_L(t)$

$$\dot{\mu}_F(t)\xi_L(t) = (\rho - [D(1-\tau(t))])\mu_F(t)\xi_L(t). \tag{A23}$$

The sum of $(A22)$ and $(A23)$ yields the differential equation

$$\frac{d(\mu_F(t)\xi_L(t))}{dt} = \rho\mu_F(t)\xi_L(t) - (\Omega_2 - t)\rho + 1 \tag{A24}$$

with the following solution

$$\mu_F(t)\,\xi_L(t) = \Omega_3 e^{\rho t} + \Omega_2 - t - \frac{2}{\rho}, \tag{A25}$$

where Ω_2, Ω_3 are integration constants.

Condition $(A13)$ with the help of $(A20), (A25)$ can be expressed as

$$\begin{aligned} \tau(t) &= \frac{1}{D\left[s(t)\,\psi_L(t) - \xi_L(t)\,\mu_F(t)\right]} \\ &= \frac{1}{D\left[\Omega_2 - t - \Omega_2 + t - \Omega_3 e^{\rho t} + \frac{2}{\rho}\right]} \\ &= \frac{\rho}{D\left(2 - \Omega_3 \rho e^{\rho t}\right)}. \end{aligned}$$

As the tax rule is always a positive number, it is reasonable to set $\Omega_3 = 0$, so the tax rule is simplified to

$$\tau = \frac{\rho}{2D}. \quad \square$$

Proof of Proposition 3

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The statement of the Nash problem is:

$$\begin{aligned} &\max \int_0^\infty e^{-\rho t}\left[\ln A_i(t) + \ln G(t)\right] dt \\ \text{s. t.} \qquad & G(t) = \sum_{k=1}^{N} \tau_k(t) D s_k(t) \\ & \dot{s}_k(t) = (1 - \tau_k(t))\, D s_k(t) - A_k(t) \\ & s_k(0) = s_{k_0}, \quad k = 1, 2, \ldots, N \end{aligned}$$

The Hamiltonian current value for the i firm is formulated as follows:

$$\begin{aligned} H_i \quad = \quad & \ln A_i + \ln\left[D\left(\tau_i(t)\,s_i(t) + \sum_{j=1, j\neq i}^{N} \tau_j(t)\,s_j(t)\right)\right] + \\ & + \lambda_{ii}(t)\left[D\left(1 - \tau_i(t)\right)s_i(t) - A_i(t)\right] + \\ & + \sum_{j=1, j\neq i}^{N} \lambda_{ij}(t)\left[D\left(1 - \tau_j(t)\right)s_j(t) - A_j(t)\right] \end{aligned} \tag{B1}$$

Taking the necessary first order conditions:

$$\frac{\partial H_i}{\partial A_i(t)} = 0 \Leftrightarrow \frac{1}{A_i(t)} = \lambda_{ii} \tag{B2}$$

$$\frac{\partial H_i}{\partial \tau_i(t)} = \frac{s_i(t)}{s_i(t)\,\tau_i(t) + \sum\limits_{j=1, j\neq i}^{N} s_j(t)\,\tau_j(t)} - \lambda_{ii}`(t)\,Ds_i(t) = 0 \tag{B3}$$

$$-\frac{\partial H_i}{\partial s_i} = \dot{\lambda}_{ii}(t) - \rho\lambda_{ii} \Leftrightarrow$$
$$\Leftrightarrow \dot{\lambda}_{ii}(t) = \lambda_{ii}(t)\,[\rho - D(1-\tau_i(t))] - \frac{\tau_i(t)}{s_i(t)\,\tau_i(t) + \sum\limits_{j=1, j\neq i}^{N} s_i(t)\,\tau_j(t)} \tag{B4}$$

$$-\frac{\partial H_i}{\partial s_j} = \dot{\lambda}_{ij}(t) - \rho\lambda_{ij} \Leftrightarrow$$
$$\Leftrightarrow \dot{\lambda}_{ij}(t) = \lambda_{ij}(t)\,[\rho - D(1-\tau_j(t))] - \frac{\tau_j(t)}{s_i(t)\,\tau_i(t) + \sum\limits_{j=1, j\neq i}^{N} s_i(t)\,\tau_j(t)} \tag{B5}$$

And the transversality condition is $\lim\limits_{t\to\infty} e^{-\rho t}\lambda_{ii}(t)\,s_i(t) = 0$.

Assuming symmetric conditions $s_i(t) = s_j(t) = s(t), \tau_i(t) = \tau_j(t) = \tau(t)$, as a consequence $\lambda_{ii}(t) = \lambda_{jj}(t) = \lambda(t)$. First order conditions now can be rewritten as:

$$A(t) = 1/\lambda(t) \tag{B6}$$

$$\tau(t) = \frac{1}{ND\lambda(t)\,s(t)} \Leftrightarrow A(t) = Ns(t)\,D\tau(t) \tag{B7}$$

$$\dot{\lambda}(t)\,[Ns(t)\,\tau(t)] = \lambda(t)\,[\rho - D(1-\tau(t))]\,[Ns(t)\,\tau(t)] - \tau(t) \tag{B8}$$

after multiplication by $s(t)$ and rearrangement, yields

$$s(t)\,\dot{\lambda}(t) = s(t)\,\lambda(t)\,[\rho - D(1-\tau(t))] - \frac{1}{N}. \tag{B9}$$

Inserting eq. $(B6)$ into the dynamic constraint $\dot{s}_k(t) = (1-\tau_k(t))\,Ds_k(t) - A_k(t)$ and multiplying by $\lambda(t)$ the result is:

$$\lambda(t)\,\dot{s}(t) = s(t)\,\lambda(t)\,[D(1-\tau(t))] - 1 \tag{B10}$$

The summation of $(B9)$ and $(B10)$ yields the equation

$$\frac{d(s(t)\,\lambda(t))}{dt} = s(t)\,\lambda(t)\,\rho - \frac{N+1}{N} \tag{B11}$$

with the following solution

$$s(t)\,\lambda(t) = \frac{(N+1)/N}{\rho} + \Omega e^{\rho t}. \tag{B12}$$

Ω is the integration constant, which is set to zero in order to fulfill the transversality condition. Consequently $(B12)$ is reduced to

$$s(t)\,\lambda(t) = \frac{N+1}{N\rho} \tag{B13}$$

and considering $(B6)$, the time path of the investment decision is written

$$A(t) = \frac{N}{N+1}\rho s(t). \tag{B14}$$

From $(B7)$ we have

$$\tau(t) = \frac{1}{ND\lambda(t)s(t)} = \frac{1}{ND\frac{N+1}{N\rho}} = \frac{\rho}{D(N+1)}. \tag{B15}$$

Substituting $(B14)$ and $(B15)$ into the state dynamics $\dot{s}(t) = (1-\tau(t))Ds(t) - A(t)$, the final equation is simplified to the following:

$$\dot{s}(t) = (D-\rho)s(t)$$

with time path solution $s(t) = s_0 e^{(D-\rho)t}$.

Consequently, the other time path is the following

$$A(t) = \rho \frac{N}{N+1} s_0 e^{(D-\rho)t}. \quad \square$$